On the design of constrained PI-like output feedback tracking controllers via robust positive invariance and bilinear programming

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***Abstract*— In this paper, we design a novel PI-like output feedback tracking controller for linear systems subject to state and input constraints. The proposed solution exploits the internal model principle to design via robust positive invariance and the Extended Farkas' Lemma, a constrained set-point tracking controller. The considered polyhedral setup results in a bilinear optimization design problem that can be solved offline, allowing to simultaneously determine the sets of admissible set-points and plant's initial conditions. The proposed solution guarantees asymptotic offset-free set-point tracking, local stability, and constraints fulfillment. Simulation results show the effectiveness of the proposed design and contrast it with an alternative scheme.**

***Index Terms*— Bilinear programming, constrained set-point tracking, PI-Control, robust positive invariance.**

I. Introduction

THE output feedback tracking control problem is crucial in the automation of different engineering systems. Examples of applications can be found in the chemical and robotic industries as well as in the development of intelligent transportation systems and self-driving cars.

For reference tracking problems, the Internal Model Principle (IMP) [1] represents a seminal result. Indeed, by considering an unconstrained feedback control system, IMP defines the conditions under which a stabilizing controller also ensures tracking. On this result, different constrained and unconstrained tracking controllers have been developed in the literature from different points of view. Existing approaches range from Proportional-Integral (PI) controllers [2] to Model Predictive Control (MPC), and Reference and Command Governor solutions [3]–[6].

PI-like controllers are particularly interesting for this work, which, among all the existing regulators, is undoubtedly the most used in the industry to solve set-point tracking problems [7]. The success of PI controllers relies upon their simplicity and, compared to MPC solutions, their minimal computational footprint for online implementation. Although PI controllers have received great attention, most existing design strategies concentrate on the constraint-free scenario, where the IMP has global validity. Consequently, when constrained setups are of interest, the IMP must be carefully used because its validity is restricted to the region where the constraints are inactive. Consequently, constraints should be considered in the design problem to ensure that the resulting control strategy is viable [8]. To simultaneously ensure constraints fulfillment and closed-loop stability, popular solutions leverage Lyapunov stability theory and the concepts of positive invariance set and contractivity, see, e.g., [8]–[14].

As long as PI-control design methods for constrained control systems are concerned, existing approaches change according to the nature of the considered plant's model. Solutions based on Algebraic Riccati Equations (ARE) or Linear Matrix Inequalities (LMI) have been developed for linear systems [15], [16]; Fuzzy Lyapunov functions have been used for nonlinear systems represented by Takagi-Sugeno (TS) models [17]; Lyapunov functions have been leveraged for Linear Parameter-Varying (LPV) systems [18]; a "quasi"-LMI approach has been designed for periodic reference signals and uncertain linear systems [19].

As a common denominator, all the above solutions deal only with symmetrical input saturation constraints and define contractive ellipsoidal (or composite ellipsoidal) invariant regions.

A. Paper's contribution and organization

Differently from the existing literature, in this paper, we exploit the concept of robust positive invariance to design a locally stabilizing PI-like tracking controller with a feedforward term capable of dealing with asymmetrical polyhedral state and input constraints. In particular, we use algebraic robust positive invariance relations to define a bilinear optimization design problem capable of simultaneously computing the PI controller parameters, the set of admissible reference signals and the polyhedral state space region where the controller is robust positively invariant [12]. In different terms, for any reference signal in the determined bound and for any initial condition inside the controller's domain, the proposed controller ensures constraint fulfillment and boundedness of the state trajectory. Moreover, steady-state offset-free set-point tracking is also guaranteed under any admissible constant reference signal.

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The present work extends the preliminary results developed in [20] in the following directions. Differently from [20], the solution proposed here *(i)* is not limited to plants having a single input and single output structure; *(ii)* the state, input and reference constraints can be asymmetric; *(iii)* the magnitude of the integral error can be taken into account in the design and treated as a design parameter; *(iv)* the obtained results are contrasted with an alternative scheme. Moreover, differently from [21]–[23], the present work considers a continuous-time setup, and it shows that the constrained reference control problem can be formulated as a bilinear programming problem. Differently from [15], [16], the proposed solution can deal with polyhedral invariant sets, and it can take into account asymmetrical and polyhedral state and input constraints.

The rest of the paper is organized as follows. In Section II, basic definitions are recalled, and the considered constrained tracking design problem is formulated. In Section III, the proposed constrained PI-like controller is designed. In Section IV, two different numerical examples are shown and contrasted with the LMI-based solution developed in [12, Sec. 8.6.2]. Finally, Section V concludes the paper with final remarks and potential future research directions.

II. Preliminaries and Problem Formulation

In this section, first, basic definitions for polyhedral sets, the Extended Farkas' Lemma, and robust positively invariant set are recalled. Then, the considered setup is presented, and the constrained set-point tracking problem is stated.

The following definitions have been adapted from [8], [12].

***Definition** 1: (Convex Polyedral Set) Any closed and convex polyhedral set $\mathcal{P}(\phi) \subseteq \mathbb{R}^n$ can be characterized by a shaping matrix $P \in \mathbb{R}^{l_p \times n}$ and a vector $\phi \in \mathbb{R}^{l_p}$, with l_p and n being positive integers, i.e.,*

$$\mathcal{P}(\phi) = \{x \in \mathbb{R}^n : Px \leq \phi\}. \tag{1}$$

Note that $\mathcal{P}(\phi)$ in (1) includes the origin as an interior point iff $\phi > 0$. In the sequel, if $\phi = \mathbf{1}_* = [1, 1, \dots, 1]^T \in \mathbb{R}^*$, the resulting polyhedral set $\mathcal{P}(\mathbf{1}_*)$ will be simply denoted as $\mathcal{P}$.

***Lemma** 1: (Extended Farkas' Lemma) Consider two polyhedral sets of $\mathbb{R}^n$ defined by $\mathcal{P}_i(\phi_i) = \{x \in \mathbb{R}^n, P_i x \leq \phi_i\}$, for $i = 1, 2$, with $P_i \in \mathbb{R}^{l_{p_i} \times n}$ and positive vectors $\phi_i \in \mathbb{R}^{l_{p_i}}$. Then, $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if there exists a non-negative matrix $Q \in \mathbb{R}^{l_{p_2} \times l_{p_1}}$ such that*

$$\begin{aligned} QP_1 &= P_2, \\ Q\phi_1 &\leq \phi_2. \end{aligned} \tag{2}$$

***Definition** 2: (Robust Positively Invariant Set) A polyhedral set $\mathcal{P}(\phi) \subseteq \mathbb{R}^n$ is said to be Robust Positively Invariant (RPI) for the system $\dot{x}(t) = f(x(t), d(t))$, $t \geq 0$, $x(t) \in \mathbb{R}^n$, $d(t) \in \Delta(\psi) \subseteq \mathbb{R}^{n_d}$, where $\Delta(\psi)$ is a compact polyhedral set, if for any initial state $x(0) \in \mathcal{P}(\phi)$, the state trajectory $x(t)$ remains bounded inside $\mathcal{P}(\phi), \forall t \geq 0$ and $\forall\, d(t) \in \Delta(\psi)$.*

***Definition** 3: (Metzler Matrix) A matrix M is Metzler type, or essentially non-negative, if $M_{ij} \geq 0, \forall i \neq j$.*

Consider a Linear Time-Invariant Continuous-Time (LTIC) system in the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ the control input vector, and $y(t) \in \mathbb{R}^p$, with $p \leq m$, the measurement vector. The system matrices (A, B, C) are of suitable dimensions, with (A, B) controllable, (C, A) observable and $rank \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + p = n_{cl}$ (i.e., the system is free from transmission zeros at the origin).

***Remark** 1: For the sake of notation clarity, in what follows, the dependency of x, y, u from t is omitted.*

The state and input vectors are assumed to be subject to the following state and input constraints

$$x \in \mathcal{X} = \{x : Xx \leq \mathbf{1}_{l_x}\},\ X \in \mathbb{R}^{l_x \times n}, \tag{4}$$

$$u \in \mathcal{U} = \{u : Uu \leq \mathbf{1}_{l_u}\},\ U \in \mathbb{R}^{l_u \times m}. \tag{5}$$

Moreover, for tracking purposes, we assume that y must track a set-point reference signal $r \in \mathbb{R}^p$, where r is bounded in an asymmetric hyperrectangle described by the set

$$\mathcal{R}(\rho) = \{r : Rr \leq \rho\},\ R = \begin{bmatrix} I_p \\ -I_p \end{bmatrix} \in \mathbb{R}^{2p \times 2p}, \tag{6}$$

where $\rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \in \mathbb{R}^{2p}$ and $\rho_i = [\rho_{i1} \ \cdots \ \rho_{ip}]^T > 0$, $i = 1, 2$.

Finally, we assume that the used tracking controller presents the following structure

$$u = Ky + K_I x_I + K_r r, \tag{7}$$

where $K, K_I, K_r \in \mathbb{R}^{m \times p}$, $x_I = \int_0^t e(\tau) d\tau \in \mathbb{R}^p$, and $e = r - y$.

***Remark** 2: Note that Ky and $K_I x_I$ define a Proportional and Integral effect, respectively, while K_r is a feedforward term used to improve the set-point response, see [7, Chapter 5]. Consequently, according to the IMP [1, Section 9.2.2], any stabilizing controller having the structure of* (7)*, guarantees asymptotic offset-free set-point tracking. However, since the considered system is subject to state and input constraints, the validity of such a result may be restricted to a bounded state space region where the constraints are inactive. Note that if the entire state vector can be measured (i.e., $y = x$), then the controller* (7) *has a state feedback term.*

The problem of interest can be stated as follows:

***Problem** 1: Consider the constrained plant's model* (3)-(5)*, the reference constraint* (6) *and the controller's structure* (7)*. Design the control gains (K, K_I, K_r) in* (7)*, the vector ρ in* (6)*, and a RPI set $\mathcal{L} \subset \mathbb{R}^{n+p}$ such that for any initial condition $x_{cl}(0) = [x^T(0)\ x_I^T(0)]^T \in \mathcal{L}$, the set-point reference r is asymptotically tracked and the constraints* (4)-(5) *are fulfilled.*

III. Proposed Solution

A solution for Problem 1 is hereafter developed resorting to polyhedral robust positive invariance arguments. In particular, first, the closed-loop dynamics of (3) under (7) are considered, and the associated constraints are characterized. Then, by resorting to proper set inclusion conditions and the Extended Farkas' Lemma, all the necessary algebraic conditions characterizing the set of admissible controller's parameters, reference's bounds and RPI sets $\mathcal{L}$ are derived (see

Proposition 1). Finally, the resulting optimization problem for control design is presented (see opt. (18)).

Under the actions of the controller (7), the closed-loop dynamics of (3) are described by the following linear system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \underbrace{\begin{bmatrix} A+BKC & BK_I \\ -C & 0 \end{bmatrix}}_{A_{cl}} \underbrace{\begin{bmatrix} x \\ x_I \end{bmatrix}}_{x_{cl}} + \underbrace{\begin{bmatrix} BK_r \\ I_p \end{bmatrix}}_{B_{cl}} r. \quad (8)$$

Moreover, the input constraint (5) can be re-written in closed-loop as the following constraint:

$$\begin{bmatrix} x_{cl} \\ r \end{bmatrix} \in \mathcal{U}_{cl} = \left\{ \begin{bmatrix} x_{cl} \\ r \end{bmatrix} : U \begin{bmatrix} KC & K_I & K_r \end{bmatrix} \begin{bmatrix} x_{cl} \\ r \end{bmatrix} \leq \mathbf{1}_{l_u} \right\}. \quad (9)$$

Since in tracking problems it is desirable to have fast tracking, the magnitude of the integral state x_I vector should be minimized. In this regard, we assume that a further optional constraint x_I can be imposed. In particular, we allow the possibility of bounding each component of x_I in the asymmetric interval $-\xi_{2j}^{-1} \leq x_{I,i} \leq \xi_{1j}^{-1}$, with $\xi_{ij} > 0$ for $i = 1, 2$ and $j = 1, \ldots, p$ or, equivalently,

$$x_I \in \mathcal{X}_I = \{x_I : X_I x_I \leq \mathbf{1}_{2p}\}, \quad (10)$$

with $X_I = \begin{bmatrix} X_{I_1} \\ -X_{I_2} \end{bmatrix} \in \mathbb{R}^{2p \times p}$, $X_{I_i} = diag\{\xi_{ij}\} \in \mathbb{R}^{p \times p}$.

***Remark** 3: X_I will be later considered as a decision matrix variable of the proposed design methodology (see* (17)*).*

Consequently, the set of state constraints acting on the closed-loop system (i.e., (4) and (10)) can be re-written as the following single constraint

$$x_{cl} \in \mathcal{X}_{cl} = \left\{ x_{cl} : X_{cl} x_{cl} \leq \mathbf{1}_{l_{x_{cl}}} \right\}, \quad (11)$$

with $X_{cl} = \begin{bmatrix} X & 0 \\ 0 & X_I \end{bmatrix} \in \mathbb{R}^{l_{x_{cl}} \times n_{cl}}$, $l_{x_{cl}} = l_x + l_{x_I}$.

To deal with the fact that the IMP is only locally valid for the considered constrained system (see Remark 2), we here pursue the idea of characterizing the RPI polyhedral set $\mathcal{L}$ for (8) where the state trajectory x_{cl} is confined and constraints (9), (11) are fulfilled for any admissible reference signal.

***Remark** 4: Note that in* (8)*, r can be interpreted as a bounded disturbance. Consequently, the RPI nature of $\mathcal{L}$ can be interpreted from Definition 2 performing the following substitutions: $\dot{x} = f(x, d) \leftarrow$ (8), $\mathcal{P}(\phi) \leftarrow \mathcal{L}$, $d \leftarrow r$, and $\Delta(\psi) \leftarrow \mathcal{R}(\rho)$.*

By describing $\mathcal{L}$ as the following polyhedral set

$$\mathcal{L} = \{x_{cl} : L x_{cl} \leq \mathbf{1}_l\}, \quad (12)$$

with $L \in \mathbb{R}^{l \times n_{cl}}$ and $rank(L) = n_{cl}$, it is possible to state Proposition 1 that defines the algebraic conditions under which the controller (7) provides a solution to Problem 1.

***Proposition** 1: Consider the closed-loop system* (8) *and the polyhedral sets* (6), (9) *and* (11)*. Assume that there exists a Metzler matrix $H \in \mathbb{R}^{l \times l}$, matrices $L \in \mathbb{R}^{l \times n_{cl}}$, $V \in \mathbb{R}^{n_{cl} \times l}$ with $l > n_{cl}$, non-negative matrices $T \in \mathbb{R}^{l_{x_{cl}} \times l}$, $Q \in \mathbb{R}^{l_u \times l}$, $H_r \in \mathbb{R}^{l \times l_r}$, $Q_r \in \mathbb{R}^{l_u \times l_r}$, and a scalar $\gamma > 0$ satisfying*

$$\begin{aligned} HL &= LA_{cl}, \\ H_r R &= LB_{cl}, \\ H\mathbf{1}_l + H_r \rho &\leq -\gamma 1_l, \end{aligned} \quad (13)$$

$$\begin{aligned} TL &= X_{cl}, \\ T\mathbf{1}_l &\leq \mathbf{1}_{l_{x_{cl}}}, \end{aligned} \quad (14)$$

$$\begin{aligned} QL &= U \begin{bmatrix} KC & K_I \end{bmatrix}, \\ Q_r R &= UK_r, \\ Q\mathbf{1}_l + Q_r \rho &\leq \mathbf{1}_{l_u}, \end{aligned} \quad (15)$$

$$VL = \mathbb{I}_{n_{cl}}. \quad (16)$$

Then, the polyhedral set $\mathcal{L}$, defined by (12)*, is robust positively invariant and such that $\mathcal{L} \subseteq \mathcal{X}_{cl}$ and $\begin{bmatrix} KC & K_I \end{bmatrix} \mathcal{L} \oplus K_r \mathcal{R}(\rho) \subseteq \mathcal{U}$, where $\oplus$ denotes the Minkowski set sum operator.* Therefore, for any initial condition $x_{cl}(0) = [x^T(0)\ x_I^T(0)]^T \in \mathcal{L}$ the output y asymptotically tracks any set-point reference $r \in \mathcal{R}(\rho)$, with corresponding closed-loop trajectories fulfilling the prescribed constraints.

Proof: The proof can be divided into four parts:

1) The existence of the Metzler type matrix H, the non-negative matrix H_r and the scalar $\gamma > 0$ verifying the conditions (13) are necessary and sufficient algebraic conditions for the robust positive invariance of the set $\mathcal{L}$, see [9], [12].

2) Since for $\mathcal{L}$ we have assumed that $l > n_{cl}$, then the existence of V such that (16) holds is equivalent to $rank(L) = n_{cl}$, the first equality in (13) can be interpreted as a generalized similarity transformation that relates the spectral sets of A_{cl} and H, given by $\sigma(A_{cl}) = \{\mu_i, i = 1, \ldots, n\}$ and $\sigma(H) = \{\omega_j, j = 1, \ldots, l\}$, respectively. Thus, in the case where $rank(L) = n_{cl} < l$, we have $\sigma(A_{cl}) \subseteq \sigma(H)$. If $\gamma > 0$, the inequality in (13) implies that the Metzler type matrix H is Hurwitz because the elements of its spectrum are such that $Re(\omega_i) \leq -\bar{\omega} \leq -\gamma$, where $-\bar{\omega}$ is necessarily a real eigenvalue belonging to $\sigma(H)$ [9]. Hence, we can conclude that A_{cl} is also Hurwitz and such that $Re(\mu_i) \leq -\gamma$.

3) According to the Extended Farkas' Lemma (see Lemma 1), the existence of the non-negative matrix T satisfying the relation (14) provides necessary and sufficient conditions under which $\mathcal{L} \subseteq \mathcal{X}_{cl}$. Likewise, applying Lemma 1, the existence of non-negative matrices Q and Q_r verifying (15) or, equivalently,

$$\begin{aligned} \begin{bmatrix} Q & Q_r \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & R \end{bmatrix} &= U \begin{bmatrix} KC & K_I & K_r \end{bmatrix}, \\ \begin{bmatrix} Q & Q_r \end{bmatrix} \begin{bmatrix} \mathbf{1}_l \\ \rho \end{bmatrix} &\leq \mathbf{1}_{l_u}, \end{aligned}$$

guarantees $\begin{bmatrix} KC & K_I \end{bmatrix} \mathcal{L} \oplus K_r \mathcal{R}(\rho) \subseteq \mathcal{U}$.

4) Finally, under the conditions (13)-(16), for any reference signal $r \in \mathcal{R}(\rho)$ and for all $x_{cl}(0) = [x^T(0)\ x_I^T(0)]^T \in \mathcal{L}$, the closed-loop state trajectory remains inside $\mathcal{L}$ while fulfilling all the prescribed state and input constraints. Consequently, the system evolves in a domain where all the constraints are inactive and the closed-loop dynamics is uniquely determined by the unconstrained linear model (8), whose state matrix A_{cl} is Schur stable. Hence, the IMP is locally valid for any $r \in \mathcal{R}(\rho)$ and for all $x_{cl}(0) = [x^T(0)\ x_I^T(0)]^T \in \mathcal{L}$, concluding the proof. □

By considering

$$\Gamma(\cdot) = (K, K_I, K_r, L, H, H_r, T, Q, Q_r, V, X_I, \gamma, \rho), \quad (17)$$

as the set of decision variables for the design of the controller (7), the algebraic relations (13)-(16) define the constraints under which the controller provides a solution to Problem 1. In particular, the resulting bilinear optimization problem can be formulated as follows

$$\begin{aligned} \underset{\Gamma(\cdot)}{\text{maximize}} \quad & \Phi(\cdot), \\ \text{subject to} \quad & (13)-(16), \\ & f_\ell(\cdot) \leq \varphi_\ell,\ \ell = 1, \ldots, \bar{\ell}. \end{aligned} \tag{18}$$

where $\Phi(\cdot)$ is the desired cost function and $f_\ell(\cdot) \leq \varphi_\ell$ are $\bar{\ell}$ auxiliary constraints instrumental to imposing limits over all the non-bounded decision variables. First, it is important to note that the opt. (18) is bilinear because it involves multiplication between decision (matrices and vector) variables. Therefore, (18) can be solved employing nonlinear optimization techniques. In this regard, the extra constraints $f_\ell(\cdot) \leq \varphi_\ell$ serve the purpose of reducing the search space and improving the numerical performance of the used nonlinear optimizer (see, for instance, [21]).

As long as the cost function $\Phi(\cdot)$ is concerned, the choice is not unique, and it depends on the user's needs. Two possible choices are:

i) $\Phi(\cdot) = \Phi_1 = \|\rho\|_1$. This choice allows us to maximize the hyperrectangle $\mathcal{R}(\rho)$ of all admissible reference signals.

ii) $\Phi(\cdot) = \Phi_2 = trace(X_{I_1} + X_{I_2})$. This choice allows us to minimize the limits of the admissible integral errors $\xi_{ij}^{-1} > 0$, for $i = 1, 2$ and $j = 1, ..., p$ (see (10)).

***Remark** 5: If the set of admissible initial conditions of the closed-loop system is known and bounded inside a polyhedral set*

$$\mathcal{X}_0 = \{x_{cl}(0) : X_0 x_{cl}(0) \leq \boldsymbol{1}_{l_{x_0}}\},\ X_0 \in \mathbb{R}^{l_{x_0} \times n_{cl}}, \tag{19}$$

then, a further constraint can be added to (18) *to impose that* $\mathcal{X}_0 \subseteq \mathcal{L}$. *From Lemma 1, such a set inclusion condition is satisfied if and only if there exists a non-negative matrix* $G \in \mathbb{R}^{l \times l_{x_0}}$*, such that,*

$$G X_0 = L, \quad G \boldsymbol{1}_{l_{x_0}} \leq \boldsymbol{1}_l. \tag{20}$$

***Remark** 6: One possible way to solve the bilinear optimization* (18) *is to resort to the nonlinear state-of-the-art solver KNITRO [24], which has been already successfully employed for similar problems in [20], [21], [23]. Note that by using KNITRO, it is possible to obtain an optimal hybrid solution that can be considered halfway between a local and a global optimum. This is obtained by applying a local solver (Interior/CG algorithm) starting from multiple initial guesses properly covering a bounded decision space. The bounds on the decision space can be found using insights about the plant's constraint bounds and with a trial and error approach. For further discussions about KNITRO and its use, see [21, Section 4.2].*

Finally, we would like to recall that the proposed solution's complexity increases with the plant's dimensions (i.e., system model, state and input constraints, reference set), and RPI set $\mathcal{L}$ *complexity (i.e., the number of inequalities* l *used to describe* $\mathcal{L}$.*). Table I provides a parametric summary of the number of variables and constraints characterizing* (18).

TABLE I
VARIABLES, EQUALITIES AND INEQUALITIES IN (18)

# of variables	$mp^3 + l(n_{cl} + l + l_r + l_{x_{cl}} + l_u)$ $+2p(l_u + p + 1) + n_{cl}^2 + 1$
# of equalities	$n_{cl}(l + l_{x_{cl}} + l_u + n_{cl}) + 2p(l + l_u)$
# of inequalities	$l + l_{x_{cl}} + l_u$

IV. Numerical Examples

In this section, the proposed tracking controller design's effectiveness is validated through simulation results obtained for two different system models. In the first example, the state vector cannot be entirely measured, and the controller's performance is evaluated for the two proposed cost functions. In the second example, a full-state feedback example scenario is considered, and the proposed controller is contrasted with the solution in [12].

For both examples, the optimization (18) has been solved using KNITRO [24], where the optimization variables have been bounded (element by element) as follows

$$\begin{aligned} H, H_r, T, Q, Q_r, G \quad & \text{in} \quad \left[0, 10^2\right], \\ L, K, K_I, K_r \quad & \text{in} \quad \left[-10^2, 10^2\right], \\ V \quad & \text{in} \quad \left[-10^3, 10^3\right]. \end{aligned}$$

***Example** 1:* Consider the following linearized model of a two-tank system (adapted from [4]):

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -0.0304 & 0.0187 \\ 0 & -0.0187 \end{bmatrix} x + \begin{bmatrix} 6.6667 \\ 10 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x, \end{aligned} \tag{21}$$

where the state and input constraints are: $-0.38 \leq x_1 \leq 0.68$, $-0.35 \leq x_2 \leq 0.65$, and $\|u\|_\infty \leq 2$. Moreover, the set of admissible initial states $\mathcal{X}_0$ is not assigned. We have solved the optimization problem (18) for both $\Phi(\cdot) = \Phi_1$ and $\Phi(\cdot) = \Phi_2$ using $l = 9$, where $l = 9$ has been experimentally found (via trial and error) as the set complexity for $\mathcal{L}$ that allows achieving a good trade-off between complexity and (sub)optimality of the solution. The resulting respective RPI sets $\mathcal{L}$, namely $\mathcal{L}_1$ and $\mathcal{L}_2$, are depicted in Fig. 1.

Moreover, Table II summarizes and compares the design results for both objective functions. In particular, ρ defines the bounds of the set of admissible reference signals $\mathcal{R}(\rho)$, X_I is the shaping matrix of the set constraining the integral error, the gains K, K_I and K_r define the control law (7), and $Vol(\mathcal{L}_i)$ is the volume of the RPI region $\mathcal{L}_i$. As expected, for the cost function Φ_1, the bounds for the admissible reference signals $\mathcal{R}$ are bigger than the ones obtained for Φ_2. However, the volume of the RPI set $\mathcal{L}_1$ is smaller than $\mathcal{L}_2$. On the other hand, by using Φ_2, it is possible to obtain a smaller integral error and faster set-point tracking. Indeed, for Φ_1, $x_I \in [-18, 20]$, while for Φ_2, $x_I \in [-11, 11]$. However, the cost to pay is a reduced size for the set of admissible reference $\mathcal{R}(\rho)$. In Fig. 1, we have depicted $\mathcal{L}_1$ and $\mathcal{L}_2$, and also reported two closed-loop trajectories, both obtained using the tracking controller associated to $\mathcal{L}_1$, i.e., for $\Phi(\cdot) = \Phi_1$. Such trajectories have been obtained starting from a zero initial condition and considering the two extreme points of the admissible reference signals, i.e., $r_1^{up} = \rho_1 = 0.6668$ (corresponding to the equilibrium state $x_{cl} = \begin{bmatrix} 0.6668 & 0.6503 & 19.9743 \end{bmatrix}^T$) and

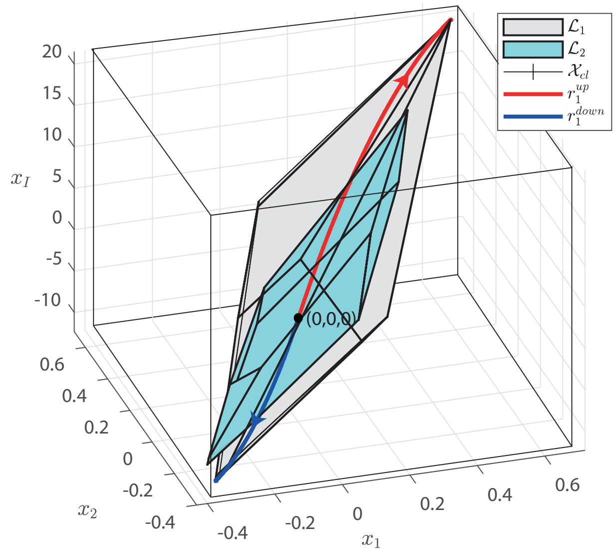

Fig. 1. RPI sets $\mathcal{L}_1$ and $\mathcal{L}_2$ and state trajectories for $r_1^{up} = \rho_1$ (−) and $r_1^{down} = -\rho_2$ (−).

$r_1^{down} = -\rho_2 = -0.3588$ (corresponding to the equilibrium state $x_{cl} = \begin{bmatrix}-0.3588 & -0.3499 & -10.7478\end{bmatrix}^T$). The obtained trajectories confirm that the designed tracking controller allows the plant to asymptotically track the assigned set-point reference signals while ensuring that the state trajectory remains confined in the constraint-admissible RPI set $\mathcal{L}_1$.

TABLE II
DESIGN RESULTS USING OPT. (18): Φ_1 VS Φ_2

Φ_i	ρ	X_I	$[K\ K_I\ K_r]^T$	$Vol(\mathcal{L}_i)$
1	$\begin{bmatrix}0.6668\\0.3588\end{bmatrix}$	$\begin{bmatrix}0.0500\\0.0551\end{bmatrix}$	$[-0.0389\ \ 0.0012\ \ 0.0053]^T$	0.2058
2	$\begin{bmatrix}0.4600\\0.3300\end{bmatrix}$	$\begin{bmatrix}0.0900\\0.0900\end{bmatrix}$	$[-0.0232\ \ 0.0006\ \ 0.0093]^T$	0.2533

Note that a standard PI controller has a structure $u = K(y - r) + K_I x_I$, which implies that $K_r = -K$. On the other hand, the proposed controller (7) contains a feedforward term with $K_r \neq -K$. To justify the proposed structure, we have tested the standard PI for both the cost functions Φ_1 and Φ_2. For Φ_1, we have obtained $\rho = [0.5397\ \ 0.2406]^T$, while for Φ_2, $X_I = [0.0398\ \ 0.0590]^T$. Such results, when compared with Table II, confirm that for Φ_1 the proposed PI with feedforward term obtains a bigger set $\mathcal{R}(\rho)$, while for Φ_2 we get a faster transient response.

***Example** 2:* Here, we compare the proposed solution with the approach developed in [12, Sec. 8.6.2], where the authors design a constrained set-point tracking controller for linear systems starting from an ellipsoidal positively and contractive control invariant region. To this end, we consider the same constrained linear system used in [12, Example 8.37].

$$\dot{x} = \begin{bmatrix}1 & 0.4\\0.8 & 0.5\end{bmatrix} x + \begin{bmatrix}0\\1\end{bmatrix} u, \qquad (22)$$

where the entire state is assumed to be measured, while the output $y = [0.2\ \ 0.1]x$ is required to track the reference signal

$$r = \begin{cases} 0.4 & \text{IF} \quad t < 150 \text{ sec}, \\ 0.2 + 0.2e^{-(t-150)/10} & \text{IF} \quad t \geq 150 \text{ sec} \end{cases} \qquad (23)$$

Moreover, the system is not subject to state constraints, but the set of admissible initial states is defined by a ball of radius 1 (i.e., $||x(0)||_2 \leq 1$). In our setup, to take into account the augmented state space vector x_{cl}, we have considered a sphere of radius one. Such a sphere has been externally approximated using an 8-face polyhedron $\mathcal{X}_{cl}$ (using the Matlab built-in function "*sphere*"). Furthermore, the control constraint is $||u||_\infty \leq 5$. Unlike the previous example, the control law (7) here considers a state feedback term, i.e., $u = Kx + K_I x_I + K_r r$.

Moreover, the optimization (18) has been configured to use as objective function Φ_2, $l = 9$, and constraints (13), (15)-(16) and (19). Furthermore, all the constraints have been assumed to be symmetric, and γ in (13) has not been considered as a variable, but fixed to $\gamma = 0.0001$ (i.e., equals to the speed of convergence in [12]).

By solving (18), the following results have been obtained: $K = \begin{bmatrix}-4.4271 & -1.9088\end{bmatrix}$, $K_I = -0.0485$, $K_r = -0.1800$, $X_I = \begin{bmatrix}0.0617 & -0.0617\end{bmatrix}^T$. Moreover, the set of admissible reference signal $\mathcal{R}(\rho)$ defines the interval $r \in [-0.4, 0.4]$ while the RPI set $\mathcal{L}$ has shaping matrix

$$L = \begin{bmatrix} 0.9014 & 0.2529 & 0.1993 \\ -0.9091 & -0.2747 & -0.0495 \\ 0.2969 & 0.0688 & 0.0267 \\ 0.5906 & 0.2436 & -0.0103 \\ -0.8984 & -0.3873 & -0.0098 \\ 0.8984 & 0.3873 & 0.0098 \\ -0.9014 & -0.2529 & -0.1993 \\ 0.9091 & 0.2747 & 0.0495 \\ -0.2083 & -0.0344 & -0.0840 \end{bmatrix}.$$

Fig. 2 shows the domain of attraction (i.e., the set of initial states from which constraint-admissible reference signals can be asymptotically approached without constraint violations during the transient) obtained by the proposed solution and the one in [12]. Note that the domain of attraction for the proposed solution is given by the projection along the $x-$coordinates of $\mathcal{L}$, namely $Proj(\mathcal{L})$, while for [12] it is given by the ellipsoidal set $\mathcal{E}(Q)$, computed via an LMI optimization. It is possible to appreciate that the domain of attraction of the proposed solution is bigger than the one obtained by [12]. Consequently, also the set of admissible reference signals is larger. This is also confirmed in Fig. 3 where the tracking capabilities of the two strategies for the piecewise reference signal (23) are compared. In particular, the proposed solution is capable of asymptotically tracking r (note that $r \in \mathcal{R}(\rho)$) while the controller in [12] is unable to track the reference for $t < 150$ sec.

V. CONCLUSION

In this paper, we have developed a PI-like set-point tracking controller for linear systems subject to polyhedral state and control constraints. The peculiar feature of the proposed design consists in the definition of a single bilinear programming problem capable of simultaneously computing the controller parameters, the set of admissible reference signals, and the

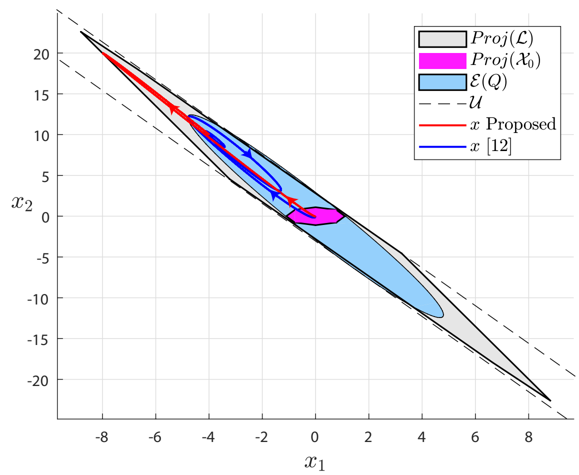

Fig. 2. Domain of attraction: proposed vs [12, Sec. 8.6.2].

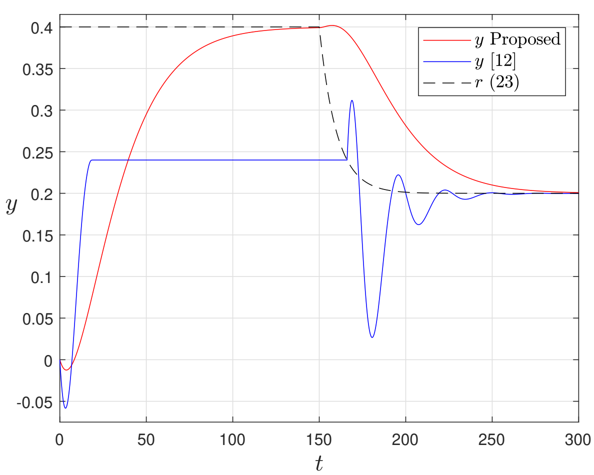

Fig. 3. Tracking performance: proposed vs [12, Sec. 8.6.2].

controller's domain of attraction. The proposed solution leveraged robust positive invariance arguments to define the algebraic conditions under which the proposed controller ensures set-point tracking and constraint fulfillment. The simulation results and comparisons with an alternative scheme have illustrated the potential of the proposed solution.

Future works will be devoted to extending the proposed approach to deal with disturbances and more complex models for the plant and reference signal. Moreover, the possibility of using the proposed controller as the terminal controller of a dual-mode model predictive controller will also be investigated.

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