Set-theoretic output feedback control: a bilinear programming approach ⋆

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Abstract

This paper addresses the problem of designing output feedback controllers for constrained linear systems subject to bounded process and measurement disturbances. The proposed solution extends the set-theoretic model predictive control framework to deal with constrained output regulation problems. In particular, this is here achieved by adequately exploiting the extended Farka's lemma to offline compute through bilinear optimization problems, a family of robust one-step controllable sets, and associated output feedback control gains. It is then formally proved that such computations can be online leveraged to design a simple switching controller capable of ensuring, by construction, that the state-trajectory of the system is always uniformly ultimately bounded, in a finite number of steps, into a small robust control invariant region. Finally, the effectiveness and benefits of the proposed solution are verified through a numerical example and compared with three alternative schemes.

Key words: Output feedback control, constrained control, set-theoretic MPC, extended Farkas's lemma, bilinear programming

1 Introduction

This paper is interested in designing robust output feedback controllers for systems subject to state and input constraints and persistent disturbances. Different solutions have been proposed by resorting to robust Model Predictive Control (MPC) frameworks. In [24], the authors propose to decouple the state feedback controller and state estimator's design and then verify that robust stability is preserved when the resulting augmented output feedback controller is considered. In [10], theoretical conditions are given to guarantee the stability of nonlinear MPC when used tighter with a state-observer. In [9], a sequence of the output feedback controller is first offline designed for different bounds on the state-estimation error set and then online used according to the error realization. In [16], a robust output feedback MPC is designed, and the robust stability test is incorporated into a linear matrix inequality (LMI) condition that is proved to be feasible under an appropriate small-gain condition. In [18], the regulation problem is solved employing a Luenberger state observe and a tube-based robust MPC. Along similar lines, in [14], the conservativeness of the resulting tube-based MPC is reduced. In [22], a tube-based MPC is also designed. However, different from other solutions, the proposed approach is independent of the used state-estimator algorithm. In [15], a simplified single tube-based robust output feedback MPC is proposed, and it is shown that its computational complexity is equal to the one that would be obtained if the full state would be available.

Among the non-MPC-based control strategies, of interest are the solutions leveraging the concept of robust control invariant sets. By considering a polyhedral framework (i.e., both the plant's constraints and disturbances are polyhedral sets), such a concept has been successfully employed to design robust state feedback control strategies, see, e.g., [3,4,11,20,23] and references therein. The control problem can be tackled either by directly imposing the set-invariant property on the constraint set or by indirectly looking for the largest invariant set contained in it. While the first approach is desirable

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(i.e., the resulting control design problem can be formulated as a Linear Programming (LP) problem), it is not generally feasible. On the other hand, the indirect approaches are more flexible but have the drawback that linear and convex programming approaches cannot be directly used. Moreover, if output feedback controllers are of interest, then the control design problem becomes even more challenging, and very few solutions have been proposed in the literature. In [8], by resorting to the concept of output feedback control-invariant sets, the design of a robustly stabilizing dynamical output feedback control is obtained by embedding the estimator in the compensator structure. It is shown that the uncertainty associated with the estimator is reduced using the contraction of the robust invariant set associated with the implicit estimator. In [6], a static output feedback controller is designed through new algebraic conditions ensuring that the state trajectory is ultimately bounded in a small robust control invariant set around the origin. The joint design of the control gain, ultimate bounded set, and $\Delta-$invariant controller's domain of attraction is obtained through a bilinear optimization approach. The authors also show the approach's ability to deal with state and dynamic-output feedback control laws.

1.1 Contribution

This paper proposes a novel set-theoretic output feedback controller to deal with the robust output regulation problem of systems subject to process and measurement disturbances and state and input constraints. The proposed scheme resorts to the joint use of the concepts of robust control invariant (RCI) set and robust one-step controllable sets (ROSC). In the state feedback case, a similar paradigm has been successfully employed to develop a computationally low-demanding MPC solution known as dual-mode set-theoretic MPC, see, e.g., [1,2,4,17]. However, to the best of the authors' knowledge, such a framework has not been used to deal with the output regulation problem. In this paper, we show that a computationally affordable switching output regulator can still be obtained in the output feedback case and that the design of such a controller can be obtained by jointly leveraging extended Farkas's lemma arguments and bilinear programming tools. The main features of the proposed controller can be summarized as follows: (i) state/input constraints, as well as plant and measurement disturbances, can be all taken into account, (ii) most of the required computations are moved into an offline phase, so leaving online the computation of simple set-membership tests, (iii) for any initial condition belonging to the controller's domain of attraction, the controller ensures that the state trajectory is uniformly ultimately bounded into a small RCI set in a finite and offline known number of steps.

The paper is organized as follows: first, by considering the class of static output feedback controllers, we geometrically characterized the RCI and ROSC set concepts, and we present the proposed switching output feedback control strategy. Then, by resorting to the extended Farkas lemma and bilinear optimization tools, we provide a detailed computable control scheme. Finally, simulation results are presented to contrast the proposed solution with existing approaches.

Notation: The sets of real numbers, real-values column vectors of dimension $n_v > 0$ and real-values matrices of dimension $n_r \times n_c$, $n_r, n_c > 0$ are denoted with $\mathbb{R}$, $\mathbb{R}^{n_c}$ and $\mathbb{R}^{n_r \times n_c}$, respectively. A non-negative matrix M is such that all its entries, namely M_{ij}, are non-negative, i.e. $M_{ij} \geq 0, \forall i,j$. The vectors $\mathbf{0}_p \in \mathbb{R}^n$, $\mathbf{1}_p \in \mathbb{R}^n$ denotes columns vectors containing only zeros or ones in all the components. Given a vector $v \in \mathbb{R}^{n_v}$, v_k represents the value of v at the discrete time instant $k \in \mathbb{Z}_+ := \{0, 1, ...\}$. Given an invertible square matrix M, M^{-1} denotes its inverse. Any closed convex polyhedral set $\mathcal{P} \in \mathbb{R}^n$, containing the origin in its interior, is represented by $\mathcal{P} = \{x \in \mathbb{R}^n : Px \leq \phi\}$, with $P \in \mathbb{R}^{l_p \times n}$ and $\phi \in \mathbb{R}^{l_p}$ a positive vector; $\mathcal{P}$ is compact (closed and bounded) $\Leftrightarrow rank(P) = n$.

2 Preliminaries

Consider the following discrete-time Linear Time-Invariant (LTI) system:

$$x_{k+1} = Ax_k + Bu_k + B_p p_k \quad (1)$$
$$y_k = Cx_k + D_\eta \eta_k \quad (2)$$

where $u_k \in \mathbb{R}^m$ is the control input, $x_k \in \mathbb{R}^n$ the state space vector, $y_k \in \mathbb{R}^p$ the output vector, and (A, B, C) are the system matrices of suitable dimensions with $rank(C) = p$. The input and the state vectors (u_k, x_k) are subject to the followings state and input constraints

$$u_k \in \mathcal{U}, \quad x_k \in \mathcal{X}, \quad \forall k \geq 0, \quad (3)$$

where $\mathcal{U} \subset \mathbb{R}^m$, $\mathcal{X} \subset \mathbb{R}^n$ are compact subsets with $\mathbf{0}_m \in \mathcal{U}$ and $\mathbf{0}_n \in \mathcal{X}$. Moreover, $p_k \in \mathbb{R}^s$, $\eta_k \in \mathbb{R}^r$ are exogenous and bounded process and measurement disturbances, with

$$p_k \in \mathcal{P}, \quad \forall k \geq 0 \quad (4)$$
$$\eta_k \in \mathcal{N}, \quad \forall k \geq 0 \quad (5)$$

and $\mathcal{P} \subset \mathbb{R}^s$ and $\mathcal{N} \subset \mathbb{R}^r$ compact subsets with $\mathbf{0}_s \in \mathcal{P}$ and $\mathbf{0}_r \in \mathcal{N}$, respectively. Without loss of generality, we assume that all the bounding sets are described as the followings polyhedra:

$$\begin{aligned}\mathcal{X} &= \{x \in \mathbb{R}^n : Xx \leq \mathbf{1}_{l_x}\},\ X \in \mathbb{R}^{l_x \times n}\\ \mathcal{U} &= \{u \in \mathbb{R}^m : Uu \leq \mathbf{1}_{l_u}\},\ U \in \mathbb{R}^{l_u \times m}\\ \mathcal{P} &= \{p \in \mathbb{R}^s : Pp \leq \mathbf{1}_{l_p}\},\ \ P \in \mathbb{R}^{l_p \times s}\\ \mathcal{N} &= \{n \in \mathbb{R}^r : N\eta \leq \mathbf{1}_{l_n}\},\ N \in \mathbb{R}^{l_n \times r}\end{aligned} \tag{6}$$

The following definitions are used along the rest of the paper:

Definition 1 *Let $\mathcal{Q} \subset \mathcal{X}$ be a region of interest. The system* (1) *under an admissible control law $u_k \in \mathcal{U}$ is said to be Uniformly Ultimately Bounded (UUB) in $\mathcal{Q}$ if for all $\mu > 0$, there exists a function $T(\mu) > 0$ such that $\forall \|x_0\| \leq \mu \rightarrow x_k \in \mathcal{Q}, \forall d_k \in \mathcal{D}$ and $\forall\, k \geq T(\mu)$ [4].*

Definition 2 *A set $\mathcal{Q} \subseteq \mathcal{X}_i$ is said Robust Control Invariant (RCI) for (1) under (3)-(4) if [5]:*

$$\forall x \in \mathcal{Q} \rightarrow \exists u \in \mathcal{U}\colon Ax + Bu + B_p p \in \mathcal{Q},\ \ \forall p \in \mathcal{P} \tag{7}$$

Definition 3 *Consider the plant model (1) under (3)-(4), and a set $\mathcal{L}_i \subset \mathcal{X}$. The set of states Robustly One-Step Controllable (ROSC) to $\mathcal{L}_i$, namely $\mathcal{L}_{i+1} \subset \mathcal{X}$, is defined as [5]:*

$$\mathcal{L}_{i+1} := \{x \in \mathcal{X} : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + B_p p \in \mathcal{L}_i,\ \forall p \in \mathcal{P}\} \tag{8}$$

Definition 4 *Given two sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^{n_s}$, their Minkowski/Pontryagin set sum ($\oplus$) and difference ($\ominus$) are [5]:*

$$\begin{aligned}\mathcal{S}_1 \oplus \mathcal{S}_2 &= \{s_1 + s_2 : s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\}\\ \mathcal{S}_1 \ominus \mathcal{S}_2 &= \{s_1 \in \mathcal{S}_1 : s_1 + s_2 \in \mathcal{S}_1, \forall s_2 \in \mathcal{S}_2\}\end{aligned}$$

The following extension of the Farkas' Lemma (see, for instance, [4, 12]) plays a fundamental role in describing the inclusion of two polyhedral sets. In this paper, the following result will be used to describe the previous definitions of RCI and ROSC sets algebraically.

Lemma 1 *Extended Farkas' Lemma (EFL): Consider two polyhedral sets of $\mathbb{R}^n$, defined by $\mathcal{P}_i = \{x \in \mathbb{R}^n : P_i x \leq \phi_i\}$, for $i = 1, 2$, with $P_i \in \mathbb{R}^{l_{p_i} \times n}$ and positive vectors $\phi_i \in \mathbb{R}^{l_{p_i}}$. Then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ or, equivalently, $P_2 x \leq \phi_2$, $\forall x : P_1 x \leq \phi_1$, if and only if there exists a non-negative matrix $Q \in \mathbb{R}^{l_{p_2} \times l_{p_1}}$ such that $QP_1 = P_2$ and $Q\phi_1 \leq \phi_2$.*

3 Problem Formulation

Consider the problem of stabilization the constrained system (1)-(5) by means of an output feedback control law

$$u_k = f(y_k) \tag{9}$$

where $f(y_k) : \mathbb{R}^p \rightarrow \mathbb{R}^m$ is a function characterizing the output control logic.

The control problem addressed in this paper can be stated as follows:

Problem 1 *Find a stabilizing output feedback control function (9) and its domain of attraction $\mathcal{L}_D \subseteq \mathcal{X}$, $0_n \in \mathcal{L}_D$ such that $\forall\, x_0 \in \mathcal{L}_D$ and persistent disturbances (4)-(5), the following properties are met:*

- *$\mathcal{L}_D$ is a RCI set for* (1)-(5) *under* (9).
- *There exist a small RCI region $\mathcal{L}_0 \subseteq \mathcal{L}_D$, $\mathbf{0}_n \in \mathcal{L}_0$ for* (1)-(5) *under* (9)*, where the state trajectory is UUB in a finite and a-priori known numbers of steps.*
- *The state and input constraints (3) are fulfilled.*

To improve as much as possible the clarity of the presentation, the rest of the manuscript is organized as follows. First, by neglecting the computational details, the proposed solution is geometrically described and its properties formally proved (Section 4). Then, all the computational aspects are presented (Section 5).

4 Proposed Solution

We propose solving *Problem* 1 by means of a switching output feedback controller exploiting set-theoretic controllability arguments. The family of switching static output feedback (SoF) controllers is offline designed as follows:

- First, by considering a single SoF control law

  $$u_k = K_0 y_k \tag{10}$$

  the gain $K_0 \in \mathbb{R}^{m \times n}$ is optimized to obtain a small RCI region $\mathcal{L}_0 \subseteq \delta\mathcal{X}, 0_n \in \mathcal{L}_0$, $0 < \delta_0 \leq 1$ for (1)-(5) satisfying the conditions (11)-(13):

  $$(A + BK_0C)\mathcal{L}_0 \oplus B_p\mathcal{P} \oplus BK_0D_\eta\mathcal{N} \subseteq \mathcal{L}_0 \tag{11}$$
  $$\mathcal{L}_0 \subseteq \delta_0\mathcal{X} \tag{12}$$
  $$K_0C\mathcal{L}_0 \oplus K_0D_\eta\mathcal{N} \subseteq \mathcal{U} \tag{13}$$

  Notice that the conditions (11)-(12) enforce that $\mathcal{L}_0$ is an admissible small RCI set under (10) and (4)-(5), and (13) guarantees that the input constraints are fulfilled regardless any admissible measurement error (5). It is worth to remark that the conditions (11)-(13) defines a bilinear optimization problem and that the related computational details are addressed in subsection 5.1.

- Second, the domain of the terminal controller, i.e. $\mathcal{L}_0$, is enlarged by recursively computing a family of robust one-step controllable (ROSC) sets, namely $\{\mathcal{L}_i\}_{i=0}^{\bar{N}}$, $\bar{N} > 0$ where each set $\mathcal{L}_i \subseteq \mathcal{X}$ is associated to a different SoF gain $K_i \in \mathbb{R}^{m\times n}$. Each pair $(\mathcal{L}_i, K_i)$ is computed to satisfy the inclusion conditions (14)-(17):

$$(A + BK_iC)\mathcal{L}_i \oplus B_p\mathcal{P} \oplus BK_iD_\eta\mathcal{N} \subseteq \mathcal{L}_{i-1} \quad (14)$$
$$\mathcal{L}_i \subseteq \mathcal{X} \quad (15)$$
$$K_iC\mathcal{L}_i \oplus K_iD_\eta\mathcal{N}\mathcal{L}_i \subseteq \mathcal{U} \quad (16)$$
$$\mathcal{L}_{i-1} \subseteq \mathcal{L}_i \quad (17)$$

while $\bar{N}$ (i.e., the number of ROSC sets computed) is constructively obtained by recursively applying (14)-(17) until the set growth saturates, i.e.,:

$$\bar{N} = \left(\min_{j\geq 1} j - 1\right) \text{ s.t. } \mathcal{L}_j = \mathcal{L}_{j-1} \quad (18)$$

Therefore, $\mathcal{L}_D := \bigcup_{i=1}^{\bar{N}} \mathcal{L}_i \subseteq \mathcal{X}$ defines the Domain of Attraction (DoA) associated to the family of SoF controllers $\{K_i\}_{i=0}^{\bar{N}}$.

Remark 1 *Please note that the set containment conditions (14)-(16) are necessary to define a robust one-step controllable set (see Definition 3) under the SoF law $u_k = K_iy_k$. On the other hand, (17) is not strictly necessary, however, it is instrumental to ensure the feasibility of the proposed output feedback controller if $rank(C) < n$ and/or $\mathcal{N} \neq \mathbf{0}_r$, see, e.g., Propositions 1. The computational details related to the bilinear optimization problem (14)-(17) are given in subsection 5.2.*

4.1 Set-theoretic with output feedback

In this subsection, we consider the case where the entire state vector cannot be measured. Without loss of generality, in what follows we assume that $C \in \mathbb{R}^{p\times n}$ and $p < n$.

Proposition 1 *Consider the plant model (1)-(5) with $p < n$, a family of SoF controllers $\{K_i, \mathcal{L}_i\}_{i=0}^{\bar{N}}$ satisfying the conditions (10)-(18) and an initial condition $x_0 \in \mathcal{L}_D$. If at each time instant the $\bar{i}_k-$th SoF controller (i.e., $u_k = K_{\bar{i}_k}y_k$) is selected according the switching rule (19)*

$$\bar{i}_k = \begin{cases} \bar{N} - k & \textbf{\textit{if}}\ k < \bar{N} \\ 0 & \textbf{\textit{otherwise}} \end{cases} \quad (19)$$

then $u_k \in \mathcal{U}$ and $x_{k+1} \in \mathcal{L}_{max(0,\, \bar{i}_k-1)}$, $\forall\, k$ and the recursive feasibility of the control strategy is guaranteed. Moreover, $\mathcal{L}_D$ is a RCI set and the state trajectory is UUB into the terminal set $\mathcal{L}_0$ in at most $\bar{N}$ steps.

Proof 1 *If $p < n$, C is not invertible and the set $\mathcal{L}_i$ containing x_0 cannot be found. On the other hand, by leveraging the nested nature of the computed ROSC sets (see condition (17)), it is always true that $x_0 \in \mathcal{L}_{\bar{N}}$. This choice, by construction, ensures that $u_0 = K_{\bar{N}}y_0$ is admissible (i.e., $u_0 \in \mathcal{U}$) and that the one-step evolution is constrained into the predecessor set $\mathcal{L}_{\bar{N}-1}$ (i.e., $x_1 \in \mathcal{L}_{\bar{N}-1}$) regardless of any admissible disturbance realization (4)-(5), see (14)-(17). As a consequence, the set $\mathcal{L}_D$ is a RCI set. Moreover, at $k = 1$, the SoF controller $u_1 = K_{\bar{N}-1}y_1$ guarantees that $u_1 \in \mathcal{U}$ and $x_2 \in \mathcal{L}_{\bar{N}-2}$. By recursively applying the same procedure for $k > 1$, i.e., by using the monotonically decreasing switching law (19), the recursive feasibility of the strategy is guaranteed [1], the RCI region $\mathcal{L}_0$ is reached in $\bar{N}$ steps, and the UUB property is fulfilled.*

4.2 Set-theoretic with noisy state feedback

In this subsection, we consider the case where the entire state vector is measured with a bounded disturbance $\eta_k \in \mathcal{N}$ (or equivalently, that the state measurements are subject to a bounded measurement noise η_k uniformly distributed in its domain $\mathcal{N}$). Without loss of generality, in what follows, we assume that $C \in \mathbb{R}^{n\times n}$ and $rank(C) = n$, which implies that C is invertible.

Proposition 2 *Consider the plant model (1)-(5) with $C \in \mathbb{R}^{n\times n}$ and invertible, a family of SoF controllers $\{K_i, \mathcal{L}_i\}_{i=0}^{\bar{N}}$ satisfying the conditions (10)-(18), the sets $\{\tilde{\mathcal{L}}_i\}_{i=0}^{\bar{N}}$, $\tilde{\mathcal{L}}_i := \mathcal{L}_i \ominus C^{-1}D_\eta\mathcal{N}$, and an initial condition $x_0 \in \mathcal{L}_D$. If at each time instant the $\underline{i}_k-$th SoF controller (i.e., $u_k = K_{\underline{i}_k}y_k$) is selected according to the switching rule (20)*

$$\underline{i}_k = \begin{cases} \min\left(J(y_k, \bar{N}),\, \bar{N}\right) & \textbf{\textit{if}}\ k = 0 \\ \min\left(J(y_k, \underline{i}_{k-1} - 1),\, \max(\underline{i}_{k-1} - 1, 0)\right) & \textbf{\textit{if}}\ k > 0 \end{cases} \quad (20)$$

$$J(y_k, i_{max}) := \min_{0\leq i\leq \underline{i}_{max}} i \;\; s.t. \;\; C^{-1}y_k \in \tilde{\mathcal{L}}_i \quad (21)$$

then $u_k \in \mathcal{U}$, $x_{k+1} \in \mathcal{L}_{max(0,\, \underline{i}_k-1)}$, $\forall\, k$, and the recursive feasibility of the control strategy is guaranteed [1]. Moreover, the state trajectory is UUB into the terminal set $\mathcal{L}_0$ in at most $\underline{i}_0$ steps.

Proof 2 *If C is invertible, we can estimate x_k from y_k as:*

$$\hat{x}_k = C^{-1}y_k \quad (22)$$

By expanding the right hand side, we obtain that

$$\hat{x}_k = C^{-1}(Cx_k + D_\eta\eta_k) = x_k + C^{-1}D_\eta\eta_k \quad (23)$$

Therefore, since $\eta_k \in \mathcal{N}$, the resulting estimation error $x_k - \hat{x}_k$ is bounded by the set $\mathcal{E}_k = -C^{-1}D_\eta\mathcal{N}$. By resorting to Minkowski set difference arguments, we can *conclude that*

$$\textbf{\textit{if}}\ \hat{x}_k \in \tilde{\mathcal{L}}_i = \mathcal{L}_i \ominus (-C^{-1}D_\eta\mathcal{N})\ \textbf{\textit{then}}\ x_k \in \mathcal{L}_i \quad (24)$$

Consequently, at each time instant, the switching index (20) prescribes the use of the SoF associated with the smallest set $\mathcal{L}_i$ ensuring that $x_k \in \mathcal{L}_i$. By following the same arguments used in the proof of Proposition 1, at each time step, the switching index $\underline{i}_k$ decreases at least by one unit. The latter is sufficient to ensure the recursive feasibility of the strategy, that $\mathcal{L}_0$ is reached in at most $\underline{i}_0$ steps and that the UUB property holds true.

Remark 2 *In the worst-case scenario, the switching signal $\underline{i}_k$ obtained by (20) is upper bounded by $\bar{i}_k$ computed as in (19), i.e. $\underline{i}_k \leq \bar{i}_k$, $\forall\, k$. As a consequence, the switching logic (19) can still be used if C is invertible. On the other hand, the logic (20), by exploiting that C is invertible, is capable of better estimating the set containing the current measurement y_k. As a consequence, with (20), a faster convergence to $\mathcal{L}_0$ is expected.* □

Remark 3 *Note that the containment condition (17) can be removed/relaxed if C is invertible and $\mathcal{N} = \mathbf{0}_r$. The main rationale is that in this case there is no ambiguity to determine (using the switching logic (20)) to which set the state x_k belongs $\forall\, k$. The problem of relaxing the condition (17) in the output feedback case, when $p < n$, is an open problem not addressed in this work.* □

4.3 Algorithm

The complete control algorithm is here summarized:

Algorithm 1 Set-Theoretic Output Feedback (ST-OF)

—*Offline (given (1)-(5))*—

1: Build a small terminal RCI region $\mathcal{L}_0$ and associated SoF gain K_0 pair $(\mathcal{L}_0, K_0)$ as in (11)-(13);
2: Build a family of ROSC sets $\{\mathcal{L}_i\}$ and associated SoF controller gains $\{K_i\}$ as in (14)-(17), until the stopping condition (18) is satisfied;
3: **if** $rank(C) = n$ **then** compute

$$\{\tilde{\mathcal{L}}_i\}_{i=0}^{\bar{N}},\ \tilde{\mathcal{L}}_i := \mathcal{L}_i \ominus C^{-1}D_\eta\mathcal{N}$$

4: **end if**
5: Store $\{K_i, \tilde{\mathcal{L}}_i^y\}_{i=0}^{\bar{N}}$ for online use.

—*Online ($\forall\, k$, $x_0 \in \mathcal{L}_D$)*—

1: Given y_k, compute i_k as follows:

$$i_k = \begin{cases} \bar{i}_k \text{ by (19)} & \textbf{if } rank(C) < n \\ \underline{i}_k^f \text{ by (19) } \textit{or} \text{ (20)} & \textbf{if } rank(C) = n \end{cases} \quad (25)$$

2: Apply $u_k = K_{i_k} y_k$;

Remark 4 *Please note that if $rank(C) = n$ and $\mathcal{N} = \mathbf{0}_r$ then the ST-OF algorithm defines a switching set-theoretic state feedback (ST-SF) controller, similar to the dual-mode solution proposed in [1]. Differently from [1], the proposed algorithm does not require the online solution of an optimization problem to compute the control action. Indeed, the controller gains K_i are offline computed along with the controllable sets $\mathcal{L}_i$.* □

5 Implementation Details

In this section, the geometric conditions (11)-(13) and (14)-(17) are translated into computable algebraic relations. From now on, we assume that the controllable sets $\mathcal{L}_i$, for $i = 0, \ldots, \bar{N}$, are compact.

According to (12), (13), (15) and (16), any candidate set $\mathcal{L}_i$ must satisfy $\mathcal{L}_i \subseteq \mathcal{X}$ and the control admissibility condition

$$x_k \in \mathcal{U}_i^x, \forall x_k \in \mathcal{L}_i \text{ and } \eta_k \in \mathcal{N}, \quad (26)$$

where, by definition,

$$\mathcal{U}_i^x = \{x_k \in \mathbb{R}^n; \eta_k \in \mathbb{R}^r : U(K_iCx_k + D_\eta\eta_k) \leq \mathbf{1}_{l_u}, \forall \eta_k \in \mathcal{N}\} \quad (27)$$

Since any admissible RCI and ROSC set must satisfy the state constraints $\mathcal{X}$, the inclusion $\mathcal{L}_i \subseteq \mathcal{X}$ must be imposed. Moreover, to provide some degrees of freedom in the shape of $\mathcal{L}_i$, we add further auxiliary constraints defined by the following closed polyhedral set $\mathcal{R}_i$,

$$\mathcal{R}_i = \{x \in \mathbb{R} : R_ix_k \leq \mathbf{1}_{r_i}\}, R_i \in \mathbb{R}^{r_i \times n} \quad (28)$$

As a consequence, $\mathcal{L}_i$ is described as

$$\mathcal{L}_i = \{x \in \mathbb{R} : L_ix_k \leq \mathbf{1}_{l_i}\}, L_i \in \mathbb{R}^{l_i \times n}, rank(L_i) = n, \quad (29)$$

where, by construction,

$$L_i = \begin{bmatrix} R_i \\ \delta_i X \end{bmatrix}, \text{ and } \mathbf{1}_{l_i} = \begin{bmatrix} \mathbf{1}_{r_i} \\ \mathbf{1}_{l_x} \end{bmatrix}, \quad (30)$$

with $l_i = r_i + l_x > n$, and $0 < \delta_i \leq 1, \forall\, i$.

Remark 5 *A necessary and sufficient condition for $\mathcal{L}_i \in \mathbb{R}^n$ to be compact, is that the associated matrix $L_i \in \mathbb{R}^{l_i \times n}$ has full-column rank and $l_i > n$. Algebraically, these properties can be described by the existence of a left-inverse matrix $J_i \in \mathbb{R}^{n \times l_i}$ such that $J_iL_i = I_n$ [21].* □

5.1 Small Robust Control Invariant Region $\mathcal{L}_0$ and K_0

In this subsection, we directly leverage the Extended Farkas' Lemma to build an admissible small RCI set from the definition (11)-(13).

The small RCI-set $\mathcal{L}_0$, can be obtained from (29)-(30) (for $i = 0$) by minimizing the real non-negative scalar $\delta_0 \leq 1$ in (12). However, to obtain well-conditioned initial solutions, we add a further set inclusion

$$\mathcal{S} \subseteq \mathcal{L}_0, \tag{31}$$

where $\mathcal{S}$ is a given compact and small shape-set defined by the polyhedron

$$\mathcal{S} = \{x \in \mathbb{R}^n : Sx \leq \mathbf{1}_{\mathbf{l_s}}\}, \; S \in \mathbb{R}^{l_s \times n}, l_s > n. \tag{32}$$

The following algebraic relations, obtained by applying the Extended Farkas' Lemma (refer to Remark 6 and to the proof of Proposition 3 for further details) describe, in a matrix form, the conditions that the pair $(\mathcal{L}_0, K_0)$ must satisfy:

- the RCI conditions (11)-(13) $\Leftrightarrow \exists$ non-negative matrices $H_0 \in \mathbb{R}^{l_0 \times l_0}$, $V_0 \in \mathbb{R}^{l_0 \times l_p}$ and $W_0 \in \mathbb{R}^{l_0 \times l_\eta}$:

$$\begin{aligned}
H_0 L_0 &= L_0(A + BK_0C) & (33)\\
V_0 P &= L_0 B_p & (34)\\
W_0 N &= L_0 B K_0 D_\eta & (35)\\
H_0 \mathbf{1}_{l_0} + V_0 \mathbf{1}_{l_p} + W_0 \mathbf{1}_{l_\eta} &\leq \mathbf{1}_{l_0} & (36)\\
M_0 L_0 &= U K_0 C & (37)\\
Z_0 N &= U K_0 D_\eta & (38)\\
M_0 \mathbf{1}_{l_0} + Z_0 \mathbf{1}_{l\eta} &\leq \mathbf{1}_{lu} & (39)
\end{aligned}$$

- the inclusion (31) $\Leftrightarrow \exists$ a non-negative matrix $T_0 \in \mathbb{R}^{l_0 \times l_s}$:

$$T_0 S = L_0, \quad T_0 \mathbf{1}_{l_s} \leq \mathbf{1}_{l_0} \tag{40}$$

- $\mathcal{L}_0$ compact (see Remark 5) $\Leftrightarrow \exists\, J_0 \in \mathbb{R}^{n \times l_0}$:

$$J_0 L_0 = I_n \tag{41}$$

Notice that the algebraic relation (33) presents two bi-linear terms involving the pair of matrix decision variables (H_0, L_0) in its left-hand side, and (L_0, K_0) in the right-hand side. This last bi-linearity also appears in (35). Likewise, the left hand side of inequality (37) has a bilinear product involving the pair (M_0, L_0). The remaining conditions are all linear with regard to the set of decision variables $\Gamma_0 = \{H_0, V_0, W_0, L_0, K_0, M_0, Z_0, T_0, J_0, \delta_0\}$. Thus, a solution for Step 1 in Algorithm 1 can be obtained from the following bi-linear optimization problem:

$$\begin{aligned}
&\underset{\Gamma_0}{\text{minimize}} && \delta_0\\
&\text{subject to} && (33) - (41),\; 0 < \delta_0 \leq 1\\
& && f_\ell(\cdot) \leq \varphi_\ell
\end{aligned} \tag{42}$$

where $f_\ell(\cdot) \leq \varphi_\ell$, for $\ell = 1, \ldots, \bar{\ell}$, are additional constraints used to reduce the decision variable space. These bounds are essential to deal with non-linear or non-convex optimization problems and promote an efficient search of the optimal solution. Please refer to section 6 for a discussion about the implementation of (42) using the nonlinear KNITRO solver.

5.2 *Family of one-step controllable sets $\mathcal{L}_i$ and K_i*

The computation of the ROSC polyhedral sets is based on the following Proposition.

Proposition 3 *Consider $\mathcal{L}_0$ obtained from (42). Then let the compact sets $\mathcal{L}_i$, for any $i = 1, \ldots, \bar{N}$, be defined by (29)-(30). Then, $\mathcal{L}_i$ is ROSC to $\mathcal{L}_{i-1}$ by SoF, and $\mathcal{L}_i \supseteq \mathcal{L}_{i-1}$, if and only if there exist $K_i \in \mathbb{R}^{m \times p}$, $L_i \in \mathbb{R}^{l_i \times n}$ and non-negative matrices $H_i \in \mathbb{R}^{l_{i-1} \times l_i}$, $V_i \in \mathbb{R}^{l_{i-1} \times l_p}$, $W_i \in \mathbb{R}^{l_{i-1} \times l_\eta}$, and $T_i \in \mathbb{R}^{l_i \times l_{i-1}}$ such that*

$$\begin{aligned}
H_i L_i &= L_{i-1}(A + BK_iC) & (43)\\
V_i P &= L_{i-1} B_p & (44)\\
W_i N &= L_{i-1} B K_i D_\eta & (45)\\
H_i \mathbf{1}_{l_i} + V_i \mathbf{1}_{l_p} + W_i \mathbf{1}_{l_\eta} &\leq \mathbf{1}_{l_{i-1}} & (46)\\
M_i L_i &= U K_i C & (47)\\
Z_i N &= U K_i D_\eta & (48)\\
M_i \mathbf{1}_{l_i} + Z_i \mathbf{1}_{l\eta} &\leq \mathbf{1}_{lu} & (49)\\
T_i L_{i-1} &= L_i & (50)\\
T_i \mathbf{1}_{l_{i-1}} &\leq \mathbf{1}_{l_i} & (51)\\
J_i L_i &= I_n & (52)
\end{aligned}$$

Proof: By definition, the set $\mathcal{L}_i$ is ROSC to $\mathcal{L}_{i-1}$ by SoF, with matrix K_i, if it satisfies (14)-(17). The conditions (14)-(15) can be equivalently described by

$$\begin{aligned}
&L_{i-1}\begin{bmatrix} A_{K_i} & B_p & BK_iD_\eta \end{bmatrix}\begin{bmatrix} x_k \\ p_k \\ \eta_k \end{bmatrix} \leq \mathbf{1}_{l_{i-1}},\\
&\forall x_k, p_k \text{ and } \eta_k : diag(L_i, P, N)\begin{bmatrix} x_k \\ p_k \\ \eta_k \end{bmatrix} \leq \begin{bmatrix} \mathbf{1}_{l_i} \\ \mathbf{1}_{l_p} \\ \mathbf{1}_{l_\eta} \end{bmatrix},
\end{aligned} \tag{53}$$

where $A_{K_i} = (A + BK_iC)$ and $diag(L_i, P, N)$ stands for the block-diagonal matrix formed from the argument matrices.

Thus, by the Extended Farkas' Lemma, (53) is equivalent to the existence of a non-negative matrix $Q_i = \begin{bmatrix} H_i & V_i & W_i \end{bmatrix} \in \Re^{l_{i-1} \times (l_i + l_p + l_\eta)}$ such that:

$$\begin{aligned}
Q_i diag(L_i, P, N) &= L_{i-1}\begin{bmatrix} A_{K_i} & B_p & BK_iD_\eta \end{bmatrix},\\
Q_i\, [\, \mathbf{1}_{l_i}^T\, \mathbf{1}_p^T\, \mathbf{1}_\eta^T\,]^T &\leq \mathbf{1}_{l_{i-1}}
\end{aligned} \tag{54}$$

which corresponds to (43)-(46). Moreover, the condition (16), which is algebraically described by (26), can be re-written in the matrix form

$$
\begin{aligned}
&U \begin{bmatrix} K_i C & D_\eta \end{bmatrix} \begin{bmatrix} x_k \\ \eta_k \end{bmatrix} \leq \mathbf{1}_{l_u}, \\
&\forall x_k \text{ and } \eta_k : diag(L_i, N) \begin{bmatrix} x_k \\ \eta_k \end{bmatrix} \leq \begin{bmatrix} \mathbf{1}_{l_i} \\ \mathbf{1}_{l_\eta} \end{bmatrix}.
\end{aligned} \tag{55}
$$

By resorting to the extended Farka's Lemma and by following the same reasoning used for (54), (55) is equivalent to the existence of non negative matrices M_i and Z_i verifying (47)-(49). Furthermore, the inclusion $\mathcal{L}_{i-1} \subseteq \mathcal{L}_i$ (see (17)), which, in a matrix form, reads

$$
L_i x_k \leq \mathbf{1}_{l_i}, \forall x_k : L_{i-1} x_k \leq \mathbf{1}_{l_{i-1}},
$$

is equivalently described (using the extended Farka's Lemma) by the existence of a non-negative matrix T_i such that conditions (50) and (51) hold true. Finally, the condition (52) imposes that the set $\mathcal{L}_i$ is compact (see Remark 5). □

Remark 6 *In the above demonstration if we consider $i = 0$ and $\mathcal{L}_{i-1} = \mathcal{L}_0$, then it is possible to prove, see [6, 19], that the algebraic relations* (33)-(41) *describe the RCI set $\mathcal{L}_0$.* □

Now, notice that the algebraic relations (43) and (47) present bi-linear terms involving the pairs of matrix decision variables (H_i, L_i) and (M_i, L_i). Furthermore, because both L_{i-1} and K_{i-1} are known in the current step i, the other *ROSC* and inclusion conditions are all linear with regard to the set of decision variables $\Gamma_i = \{H_i, V_i, W_i, M_i, Z_i, L_i, K_i, T_i, J_i, \delta_i\}$. In order to maximize the size of $\mathcal{L}_i$, we introduce the following auxiliary inequalities

$$
\gamma_{t,i} L_i v_t \leq \mathbf{1}_{l_i}, t = 1, \ldots, \bar{t}. \tag{56}
$$

where $\gamma_{t,i}$, for $t = 1, \ldots, \bar{t}$, are real positive scaling factors associated to a given set $\mathcal{V}_i$ of $\bar{t} > 0$ directions $v_t \in \mathbb{R}^n$, where

$$
\mathcal{V}_i = \{\gamma_{t,i} v_t, t = 1, \ldots, \bar{t}\}. \tag{57}
$$

Hence, the computation of the ROSC set $\mathcal{L}_i$ and associated SoF matrix K_i can be obtained from the following bi-linear optimization program

$$
\begin{aligned}
\underset{\Gamma_i, \gamma_{t,i}}{\text{maximize}} \quad & \mathcal{J}_i = \sum_{t=1}^{n} \gamma_{t,i} \\
\text{subject to} \quad & (43) - (52) \text{ and } (56), \\
& \delta_{i-1} < \delta_i \leq 1, \\
& f_\ell(\cdot) \leq \varphi_\ell.
\end{aligned} \tag{58}
$$

Remark 7 *Note that the condition* (56) *imposes that $\mathcal{V}_i \subseteq \mathcal{L}_i$. Moreover, at each step i, the objective function $\mathcal{J}_i = \sum_{t=1}^{n} \gamma_t$ is used to maximize the size of $\mathcal{L}_i$ w.r.t. the set of directions in $\mathcal{V}_i$. Notice that the set $\mathcal{L}_i$ obtained from* (58) *depends on the used directions, which are designer's choices. Moreover, the criterion*

$$
\mathcal{J}_i - \mathcal{J}_{i-1} = \sum_{t=1}^{n} (\gamma_{t,i} - \gamma_{t,i-1}) \leq tol. \tag{59}
$$

with tol a small tolerance value, can be used to numerically approximate the stopping condition (18) used by the ST-OF offline algorithm. In simpler terms, if at the iteration $i > 0$ the condition (59) is verified, then $\bar{N} = i - 1$.

Remark 8 *In the proposed solution, the computation of the RCI and ROSC sets is obtained by resorting to the bilinear optimization problems* (42) *and* (58)*, respectively. Consequently, the complexity of the provided solution increases with the dimensions of the plant (i.e., system model, state and input constraints, and disturbances). For the sake of completeness, Table 1 summarizes the number of variables and constraints (equalities and inequalities) used in the proposed RCI and ROSC optimization problems.* □

6 Numerical Examples

In this section, some numerical results are presented to verify the proposed control strategy's effectiveness and compare it with three existing approaches. The simulations have been performed on Matlab 2019b, using a Windows 10 PC equipped with an AMD Ryzen 5 3600 6-Core Processor (3.59 GHz) and 16,0 GB of RAM.

6.1 Example 1

In particular, we consider a LTI system (1) defined by the matrices

$$
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, B_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
$$

The state and control constraints, and the disturbance limits in (6) are shaped by the matrices

$$
X = \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, U = \begin{bmatrix} 1.25 \\ -1 \end{bmatrix}, P = \begin{bmatrix} 10 \\ -10 \end{bmatrix}, N = \begin{bmatrix} 10 \\ -10 \end{bmatrix},
$$

corresponding to $-1 \leq x_1 \leq 1.25$, $|x_2| \leq 1$, $-1 \leq u \leq 0.8$, $|p| \leq 0.1$, and $|\eta| \leq 0.1$, respectively. The matrices

	RCI set $\mathcal{L}_0$	ROSC sets $\mathcal{L}_i$
# of Variables	$mp + l_0(n^2 + l_0 + l_p + l_n + l_u + l_s) + l_u l_n$	$mp + l_{i-1}(n + l_i^2 + l_p + l_n) + l_u(l_i + l_n) + nl_i$
# of Equalities	$l_0(n^2 + s + r) + l_u(n + r) + n^2$	$l_{i-1}(n + s + r) + l_u(n + r) + l_i n + n^2$
# of Inequalities	$l_0 + l_u + l_s$	$l_{i-1} + l_u + l_i$

Table 1
Number of variables, equality and inequality constraints used in the RCI and ROSC optimization problems.

C and D_η related to output equation (2) will be defined in the sequel, depending if the used control law is ST-SF or ST-OF.

As in [6], the efficient KNITRO solver [7] has been used to solve the required bilinear optimizations. In particular, in the optimization problems (42) and (58), f_ℓ and φ_ℓ are tuned such that each element of the positive definite variables is bounded in the interval [0, 100], each element of R, K is in $[-100, 100]$, and each element of J is in $[-1000, 1000]$. Note that by using KNITRO, we have obtained an optimal hybrid solution that can be considered halfway between a local and a global optimum. This is obtained by applying a local solver (Interior/CG algorithm) starting from multiple initial guesses properly covering the decision space.

6.1.1 State feedback controller

In this subsection, we assume a scenario where the entire state vector can be measured with some bounded errors, i.e., $C = I_2$ and $D_\eta = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Therefore, in this particular setup, the proposed strategy is a ST-SF controller (see Remark 4).

The ST-SF controller is offline designed considering $\bar{t} = 4$ auxiliary constraints (56), where the normalized vectors v_t point towards the vertices of the state constraints set $\mathcal{X}$. Furthermore, the auxiliary polyhedral sets R_i in (28) are such that $r_i = 3, \forall\, i$. By following the offline steps indicated in Algorithm 1, $\bar{N} = 6$ pairs $(K_i, \mathcal{L}_i)$ have been computed, see Table 2. Moreover, in the table, it is possible to note that the cost function $\mathcal{J}_i$ and area of the polyhedral sets, namely "$\mathcal{L}_i$ area," monotonically increase with i while the $\|K_i\|_\infty$ decreases. In Fig. 1, the obtained ROSC sets $\mathcal{L}_i$ are plotted, showing that their union covers the entire admissible state constraint regions $\mathcal{X}$. Moreover, in the same figure, we have shown (using a black dashed dot (-.) line) the plant's state trajectory evolution applying the proposed controller (using the switching rule (20)) starting from an initial condition $x_0 = [1.25, \ -1]^T \in \mathcal{L}_6$ (i.e., $i_0 = 6$). The obtained results show that in 5-step the state enters the terminal RPI region, i.e., $x_5 \in \mathcal{L}_0$, where it remains confined despite the presence of process and measurement bounded noises. Such a result is compatible with the theoretical worst-case convergence time equals to $i_0 = 6$ (see Proposition 2).

Step i	K_i	$\mathcal{L}_i$ Area	$\mathcal{J}_i$
0	[-0.4805 -0.5000]	0.2584	1.2887
1	[-0.4967 -0.4966]	0.7330	2.6329
2	[-0.8407 -0.4966]	1.8310	3.4379
3	[-0.5032 -0.1955]	2.9355	4.4746
4	[-0.2088 -0.4665]	3.7965	5.0647
5	[-0.2250 -0.3861]	4.4456	5.8774
6	[-0.1772 -0.4914]	4.5000	6.0299

Table 2
ST-OF, offline design for $r_i = 3$ and $\bar{t} = 4$.

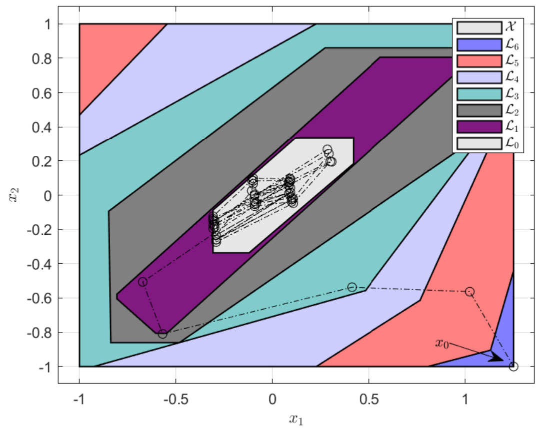

Fig. 1. ST-SF: ROSC sets and state trajectory evolution from $x_0 = [1.25, \ -1]^T$.

6.1.2 State feedback: comparison

Here, the ST-SF controller performance are compared with the dual-mode state feedback controller proposed in [1]. To provide a fair comparison, here we assume the absence of measurement noise ($\eta_k = 0, \forall\, k$ as assumed in [1]). Moreover for [1], exact polyhedral robust one-step controllable sets are computed [5]. Furthemore, the same terminal RPI set $\mathcal{L}_0$ is used in the offline computations of both strategies, and, the online simulation has been repeated 1000 times considering different noise realizations, and different initial states x_0 belonging to the two outermost ROSC sets.

The obtained results are summarized in Table 3. First, it is possible to note that a slightly smaller number of ROSC sets is computed in [1] to cover $\mathcal{X}$. The latter finds

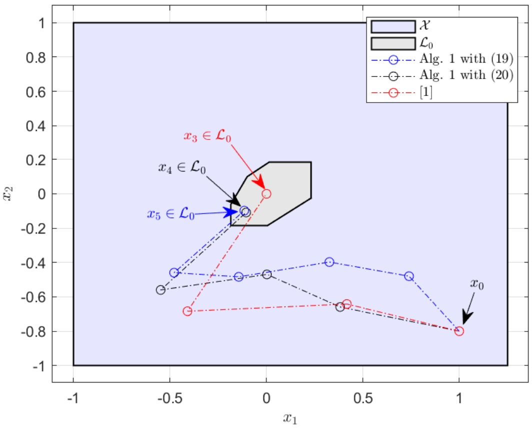

Fig. 2. State trajectory for $x_0 = [1, -0.8]^T$, under state feedback control, and until the RCI region $\mathcal{L}_0$ is reached: proposed solution vs [1].

	Avg. CPU time [s] until $x_k \in \mathcal{L}_0$	$\bar{N}$	Avg. steps to $\mathcal{L}_0$
Alg. 1 with (19)	5.9770e-07	6	5.00
Alg. 1 with (20)	2.8134e-05	6	3.90
[1]	0.05s	5	3.04

Table 3
State feedback control: proposed solution vs [1].

justification in the fact that for [1] we have computed, at each step, the biggest controllable sets, while in the proposed approach, we impose the further constraint that $\mathcal{L}_i$ is an admissible ROSC set if and only if each $x \in \mathcal{L}_i$ is one-step controllable to $\mathcal{L}_{i-1}$ under the same admissible control gain K_i. This also justifies why in our solution, the converge of x_k to $\mathcal{L}_0$, requires, in average a slightly bigger number of steps (5 using (19), 3.90 using (20), and 3.04 in [1]). This can also be noted in Fig. 2, where the state trajectory for a single initial condition is reported. On the other hand, the proposed solution outperforms [1] in terms of average computation time needed to online compute the control action. The latter is justified by the fact that contrary to [1], the proposed approach offline computes also the controller gain K_i, so making the strategy better suited for strict-real time control system contexts or where limited computational capabilities are available. Last but not least, differently from our approach, the solution in [1] cannot be used if the entire state vector cannot be measured.

6.1.3 Output feedback controller

In this subsection, we assume that only a noisy version of the first state x_1 can be measured. In particular, $C = [1\,0]$ and $D_\eta = 1$. To build the RCI and ROSC sets, we have used a configuration with more degrees of freedom by choosing $(\bar{t} = 8, r_i = 4)$. This setup, with respect to

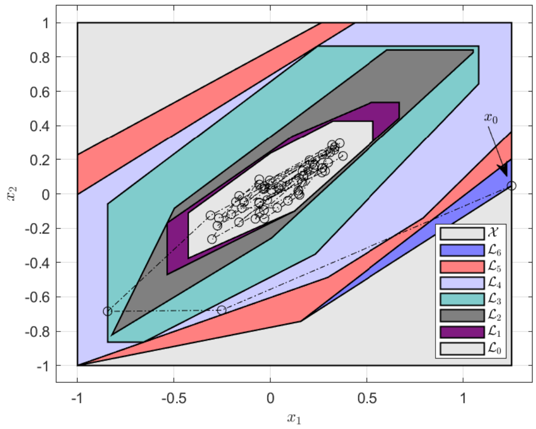

Fig. 3. ST-OF: DoA for $(\bar{t} = 8,\ r_i = 4)$, and state trajectory for $x_0 = [1.25,\ 0.047]^T \in \mathcal{L}_6$.

Step i	K_i	$\mathcal{L}_i$ Area	$\mathcal{J}_i$
0	[-0.7500]	0.3120	2.3466
1	[-0.7803]	0.4054	2.7227
2	[-0.8686]	0.7949	4.1941
3	[-0.8476]	1.7372	5.9305
4	[-0.6979]	2.5962	7.3990
5	[-0.6111]	3.0652	8.0587
6	[-0.6111]	3.1510	8.2741

Table 4
ST-OF, offline design for $(\bar{t} = 8, r_i = 4)$.

the state-feedback case, contains four more additional normalized vectors v_i, each orthogonal to one face of $\mathcal{X}$, and one more face in $\mathcal{R}_i$. In Table 4, a summary of the results for the case $(\bar{t} = 8, r_i = 4)$ is reported, where the stopping condition (59) is reached for $\bar{N} = 6$ when the ROSC sets do not entirely cover $\mathcal{X}$, see, e.g., the DoA shown in Fig. 3. Finally, the black dashed dot (-.) line in Fig. 3, representing the state-trajectory under ST-SOF (for $x_0 = [1.25,\ -1]^T \in \mathcal{L}_8$), confirms that the proposed output controller can robustly steer, is a finite number of steps (by design ≤ 8), the state vector inside the RPI region $\mathcal{L}_0$. By observing the results in Tables 2 and 4, it is possible to note that in the state feedback case, the DoA of the controller covers the whole state-constraint region; whereas, in the output feedback case, the DoA is only a subset.

6.1.4 Output feedback: comparison

We contrasted our approach with the output feedback controllers proposed in [6, 8]. First, it is worth mentioning that all the strategies share the same idea of computing the DoA offline. However, in [6], the DoA is associated with a single control gain, while in [8], an LP optimization problem is used to online compute the control inputs.

	K	DoA Area	$\mathcal{L}_0$ Area
[6] for max DoA	[-0.6111]	3.1510	3.1510
[6] for min $\mathcal{L}_0$	[-0.7500]	2.1698	0.1600
[8]	-	2.5962	-

Table 5
ST-OF: comparative results

Table 5 summarizes some relevant facts about the control solutions in [6, 8]. First, since the solution in [6] resorts to a single controller gain, a trade-off exists between the size of the terminal RCI set and the DoA of the controller. This is shown in the first two rows of Table 5 where [6] is tuned to either maximize the DoA or minimize the RCI set $\mathcal{L}_0$. In the first case, the area of the DoA and the control gain K are exactly equal to the ones computed by our proposed strategy (see Table 6). However, the strategy fails to find a smaller RCI set $\mathcal{L}_0$. In the second case, the area of the RCI set $\mathcal{L}_0$ is 95% smaller than the one computed by our solution, but the estimated DoA is 45% smaller.On the other hand, the last line of the table shows the domain of attraction computed by resorting to the methodology described in [8, section III]. In this case, the obtained DoA is equal to $\mathcal{L}_4$ computed by our approach. Consequently, the DoA of [8] is contained inside the DoA of the proposed solution and 21.4% smaller. Moreover, [8] does not compute any inner RCI set $\mathcal{L}_0$.

6.2 Example 2

In this subsection, we apply the proposed solution to a Multi-Input-Multi-Output (MIMO) system. As benchmark system, we consider the continuous-time model of a four-tank system [13], whose parameters, state and control constraints are borrowed from [25]. The corresponding discrete-time system, obtained via ZOH-discretization with sampling period $T_s = 5$ seconds, has the following system's matrices:

$$A = \begin{bmatrix} 0.9705 & 0 & 0.0207 & 0 \\ 0 & 0.9663 & 0 & 0.0195 \\ 0 & 0 & 0.9790 & 0 \\ 0 & 0 & 0 & 0.9802 \end{bmatrix},$$

$$B = \begin{bmatrix} 24.6291 & 0.5213 \\ 0.5737 & 32.7684 \\ 0 & 49.4735 \\ 57.7531 & 0 \end{bmatrix}, \; C^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The state and control constraints are $-0.45 \leq x_1 \leq 0.71$, $-0.46 \leq x_2 \leq 0.7$, $-0.45 \leq x_3 \leq 0.65$, $-0.46 \leq x_4 \leq 0.64$, $|u_1| \leq 0.4528 \times 10^{-3}$, $|u_2| \leq 0.5556 \times 10^{-3}$. The system is subject to a process disturbance acting on the control inputs, with $B_p = B$ and $|p_1| \leq 0.1132 \times 10^{-4}$, $|p_2| \leq 0.1389 \times 10^{-4}$. Moreover, the output measurements are noiseless, i.e., $D_\eta = \mathbf{0}$.

Step i	$\mathcal{L}_i$ hyper-volume	$\mathcal{L}_i$ hyper-volume % of $\mathcal{X}$
0	1.1120	68.3
1	1.1808	72.5
2	1.2604	77.4
3	1.3480	82.8
4	1.4271	87.6
5	1.5025	92.3
6	1.6003	98.3
7	1.6271	99.9

Table 6
ST-OF, offline design for $(\bar{t} = 16, r_i = 12)$.

The numerical results shown in Table 6 have been obtained for $(\bar{t} = 16, r = 12)$, where the chosen 16 directions for $\mathcal{V}_i$ corresponds to the vertices of the state-constraint set $\mathcal{X}$. The computed RCI set $\mathcal{L}_0$ presents a hyper-volume of 1.1120 that is 68.3% of $\mathcal{X}$. On the other hand, the latest ROSC set, namely $\mathcal{L}_7$, presents a hyper-volume of 1.6271 that is 99.9% of $\mathcal{X}$. Such results imply that the DoA of the proposed controller is almost equal to the admissible region $\mathcal{X}$ and that the proposed controller will confine, in at most 7 steps, the state trajectory within an RCI region $\mathcal{L}_0$ that is roughly 32% smaller than the DoA. For the same MIMO system, the strategy in [6], which considers a single constant control gain, obtains a DoA equal to $\mathcal{X}$, but the RCI set has a hyper-volume of 1.3542 that is 21.8% bigger than the one obtained with the proposed switching controller.

7 Conclusion

This paper has presented a novel output feedback controller for constrained linear systems subject to bounded process and measurement noises. The proposed solution has exploited the extended Farkas' lemma, controllability, and set invariance arguments to offline design a family of static output feedback control gains and associated domains of attraction. Online admissible and robust control actions have been computed by resorting to a switching policy defined on the offline determined gains. Given the absence of online optimizations, the proposed strategy is well-suited for strict real-time applications. Moreover, as one of its main features, it can be formally proved that the proposed controller guarantees that the state trajectory is always uniformly ultimately bounded, in a finite number of steps, into a small control invariant region. A promising research direction to obtain less conservative output feedback solutions points to switched dynamic controllers, whose structure may be inspired by [6].

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