MECH370: Modelling, Simulation and Analysis of Physical Systems

Chapter 2

Modeling of Translational Mechanical System

- Elements and element laws of translational mechanical systems
- Free body diagram (FBD)
- Interconnection laws
- Obtaining the system model

Course Outline

1. Definition and classification of dynamic systems (chapter 1)
2. *Translational mechanical systems (chapter 2)*
3. Standard forms for system models (chapter 3)
4. Block diagrams and computer simulation with Matlab/Simulink (chapter 4)
5. Rotational mechanical systems (chapter 5)
6. Electrical systems (chapter 6)
7. Analysis and solution techniques for linear systems (chapters 7 and 8)
8. Developing a linear model (chapter 9)
9. Electromechanical systems (chapter 10)
10. Thermal and fluid systems (chapters 11, 12)

Why Mathematical Models are Needed?

Review

- Analogous Systems
  - Can have the same mathematical model for different types of physical systems
  - Common analysis methods and tools can be used

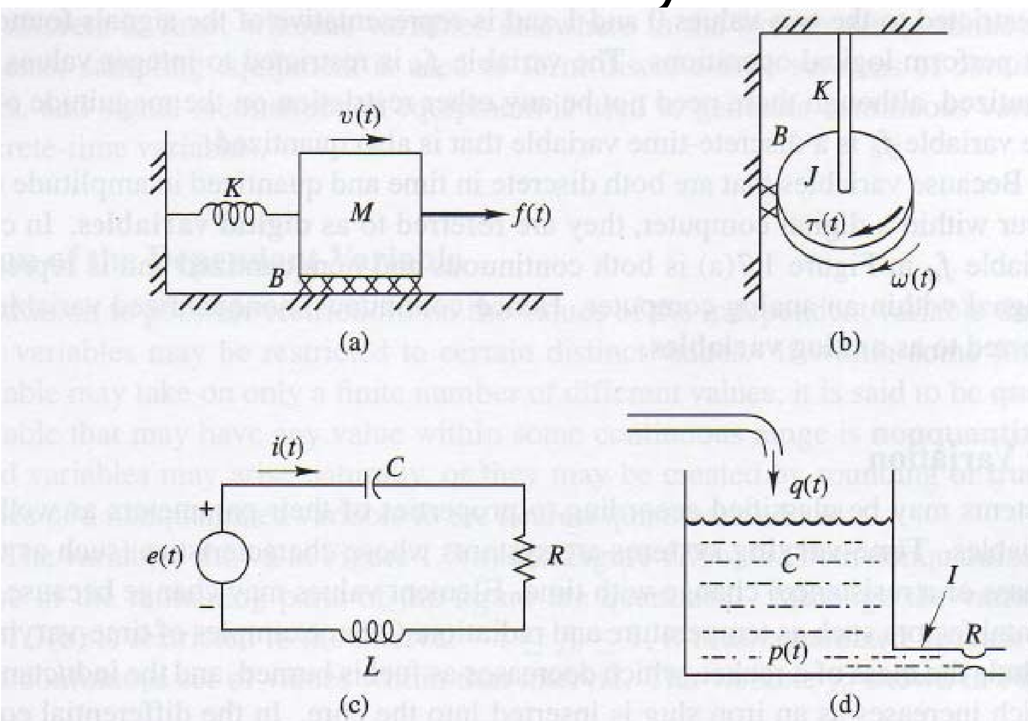


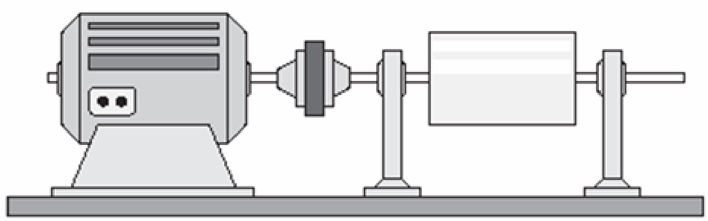
Figure 1.8 Analogous systems. (a) Translational mechanical. (b) Rotational mechanical. (c) Electrical. (d) Hydraulic.

$$
\begin{aligned}
M\frac{dv}{dt} + Bv(t) + K\int_0^t v(\lambda)d\lambda &= f(t) \\
J\frac{d\omega}{dt} + B\omega(t) + K\int_0^t \omega(\lambda)d\lambda &= \tau(t) \\
L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\lambda)d\lambda &= e(t) \\
C\frac{dp}{dt} + \frac{1}{R}p(t) + \frac{1}{I}\int_0^t p(\lambda)d\lambda &= q(t)
\end{aligned}
$$

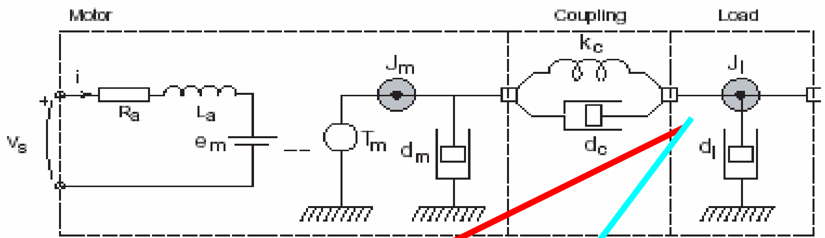
Procedure of System Modeling

Review

- Divide the system into idealized components
- Apply physical laws to the elements
- Apply interconnection laws between elements
- Combine the equations to obtain the model



More detailed schematic



State equations

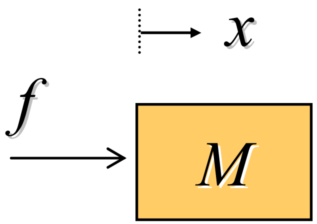
$$L_a \frac{di}{dt} = V_s - R_a i - k_m \omega_m$$

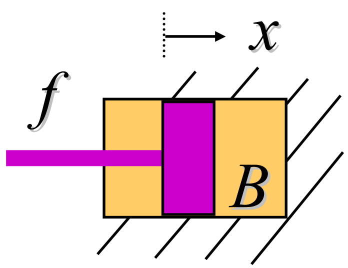
$$\frac{d\theta_m}{dt} = \omega_m \qquad J_m \frac{d\omega_m}{dt} = k_m i - d_m \omega_m - k_c(\theta_m - \theta_l) - d_c(\omega_m - \omega_c)$$

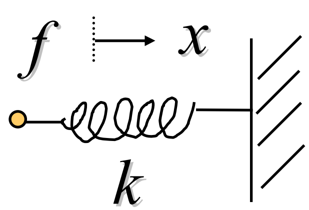
$$\frac{d\theta_l}{dt} = \omega_l \qquad J_l \frac{d\omega_l}{dt} = -d_l \omega_l - k_c(\theta_l - \theta_m) - d_c(\omega_l - \omega_m)$$

Overview of Element Models in Physical Systems

Mechanical Translational Models



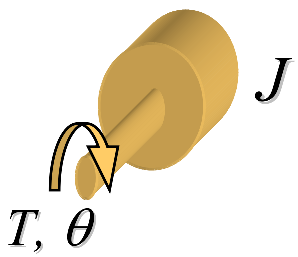


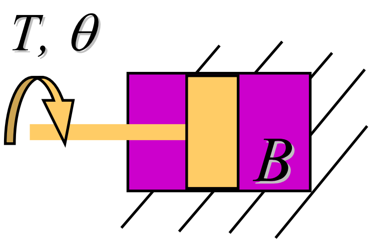


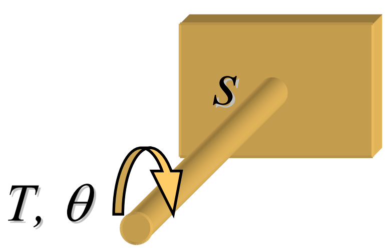
	force/velocity	force/position
Mass	$f = M\, dv/dt$	$f = M\, dx^2/dt^2$
Damper (Viscous friction)	$f = B\, v$	$f = B\, dx/dt$
Spring (Stiffness)	$f = k \int v\, dt$	$f = k\, x$

Overview of Element Models in Physical Systems

Mechanical Rotational Models



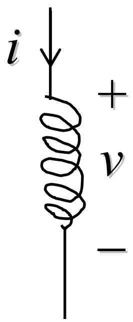


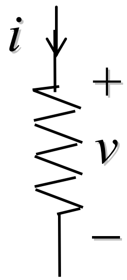


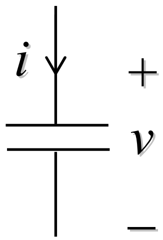
	torque/velocity	torque/position
Inertia	$T = J\, d\omega/dt$	$T = J\, d\theta^2/dt^2$
Viscous friction	$T = B\ \omega$	$T = B\, d\theta/dt$
Stiffness	$T = s \int \omega\, dt$	$T = s\ \ \theta$

Overview of Element Models in Physical Systems

Electrical Component Models



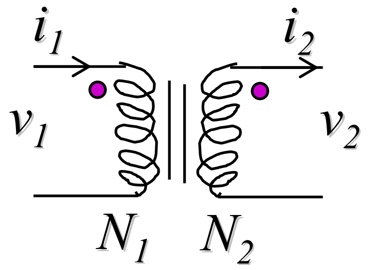




	voltage/current	voltage/charge
Inductance	$v = L\, di/dt$	$v = L\, dq^2/dt^2$
Resistance	$v = R\, i$	$v = R\, dq/dt$
Capacitance	$v = 1/C \int i\, dt$	$v = 1/C\, q$

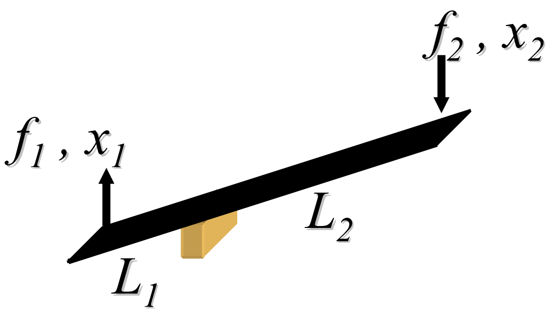
Overview of Element Models in Physical Systems

Transformation Models



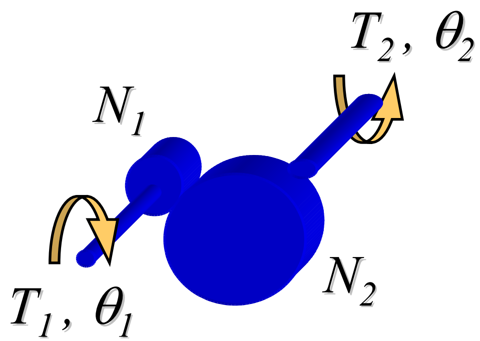
Transformer

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} \qquad \frac{i_1}{i_2} = \frac{N_2}{N_1}$$



Lever

$$\frac{f_1}{f_2} = \frac{L_2}{L_1} \qquad \frac{x_1}{x_2} = \frac{L_1}{L_2}$$



Gears

$$\frac{T_1}{T_2} = \frac{N_1}{N_2} \qquad \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

Mathematical Modelling of Mechanical Systems

Elementary parts

- A means for storing kinetic energy (mass or inertia)
- A means for storing potential energy (spring or elasticity)
- A means by which energy is gradually dissipated (damper)

Mathematical Modelling of Mechanical Systems

Motion in mechanical systems can be

- Translational
- Rotational, or
- Combination of above

Mechanical systems can be of two types

- Translational systems
- Rotational systems

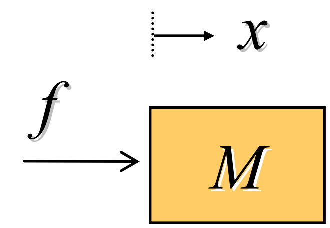
Variables that describe motion

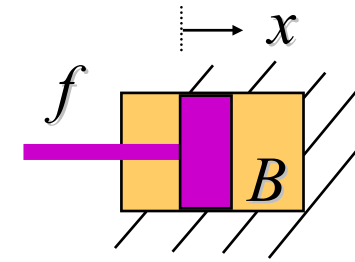
- Displacement, x
- Velocity, v
- Acceleration, a

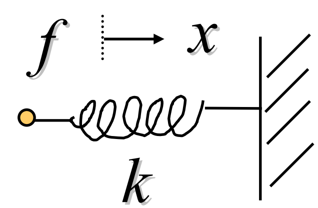
Modeling of translational mechanical systems

Variables

- x: displacement (m)
- v: velocity (m/sec) $\quad v = \dot{x} = \dfrac{d}{dt}x$
- a: acceleration (m/sec^2) $\quad a = \dfrac{dv}{dt} = \dfrac{d}{dt}(\dfrac{d}{dt}x) = \ddot{x}$
- f: force (N)
- P: power (Nm/sec, Watt) $\quad P = f \cdot v = f \cdot \dot{x} = \dfrac{dW}{dt}$
- W: work (energy) (Nm, J) $\quad W(t_1) = W(t_0) + \int_{t_0}^{t_1} P(t)dt$







All these variables are functions of time, t.

Power is defined to be the rate at which energy is supplied or dissipated.

Modeling of translational mechanical systems

Variables (cont'd)

The energy supplied between time t_o and t_1 is

$$\int_{t_0}^{t_1} P(t)dt$$

The total energy supplied from time t_0 up to any later time is

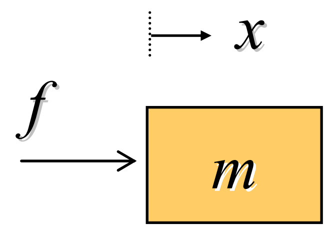
$$W(t_1) = W(t_0) + \int_{t_0}^{t_1} P(t)dt$$

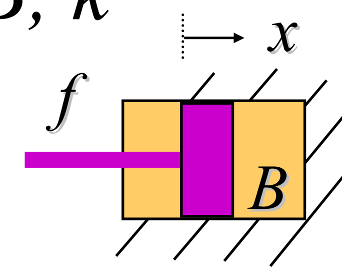
1 Nm (Newton-Meter) = 1 J (Joule)
1 Watt = 1 Joule per Second

Modeling of translational mechanical systems

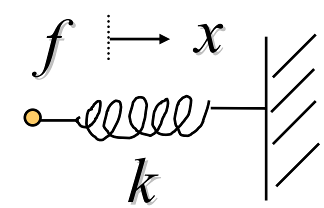
Elements

- **Three primary elements of interest**
  - – Mass (inertia) m (or M)
  - – Stiffness (spring) k
  - – Friction - Dissipation (damper) B
  - – Usually we deal with "equivalent" m, B, k
    - ➢ Distributed mass -> lumped mass





- **Lumped parameters**
  - – Mass maintains motion
  - – Stiffness restores motion
  - – Damping eliminates motion

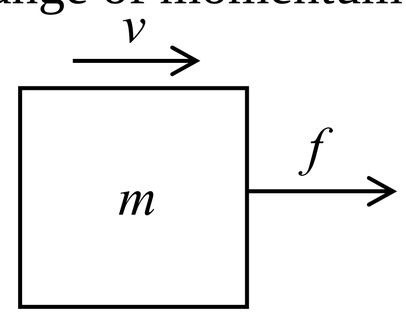


Elements

Mass

Newton's second law:
sum of forces acting on a body = time rate of change of momentum

$$f = \frac{d}{dt}(mv), \quad \text{if } \frac{dm}{dt} = 0 \Rightarrow f = m\frac{dv}{dt} = ma$$



Assumptions:

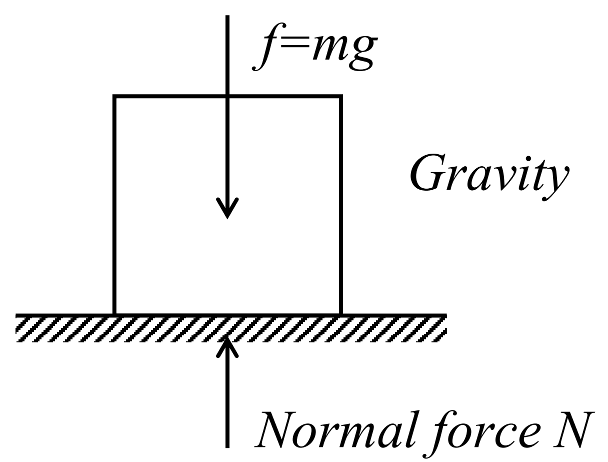
1. motions defined with respect to inertial reference frame
2. scalar quantities (1 degree of freedom)

Energy

Kinetic $E_K = \frac{1}{2}mv^2$

Potential $E_P = \frac{1}{2}mgh$

Initial conditions $v_0,\ h_0$

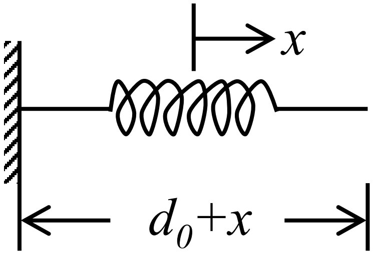


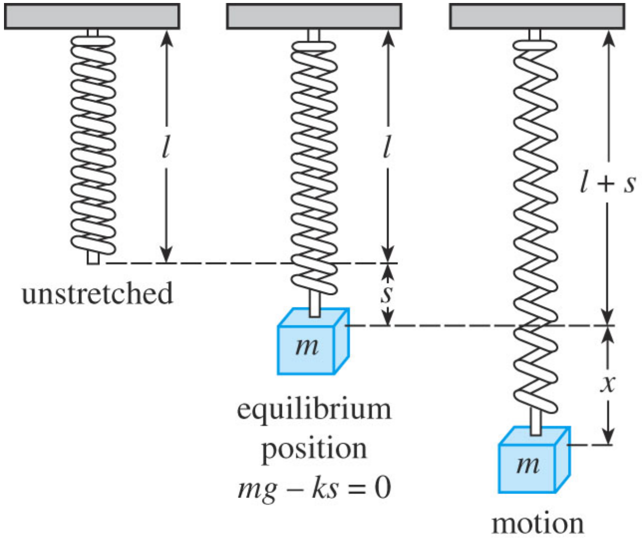
Elements (cont'd)

Stiffness (N/m)

Stiffness is the resistance of an elastic body to deflection or deformation by an applied force

Most common: ideal spring





d_0 = length of spring when no force is applied

x = elongation caused by f

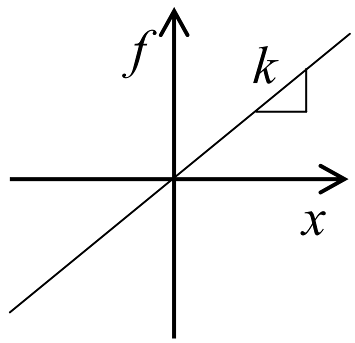
$$d(t) = d_0 + x(t) \quad \Rightarrow \quad x(t) = d(t) - d_0$$

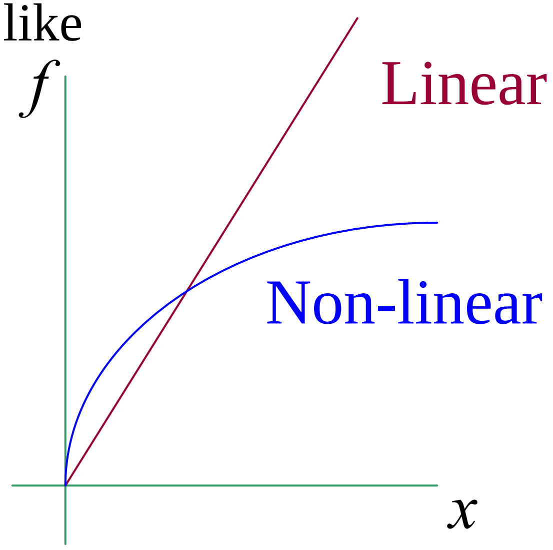
$$f = kx$$

k: stiffness constant

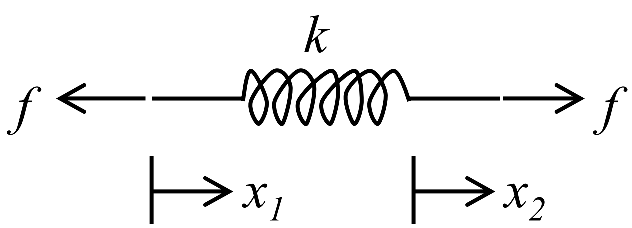
Elements (cont'd)

The linear spring is an approximation of something like





Multiple applied forces: $f = k(x_2 - x_1) = k\Delta x$

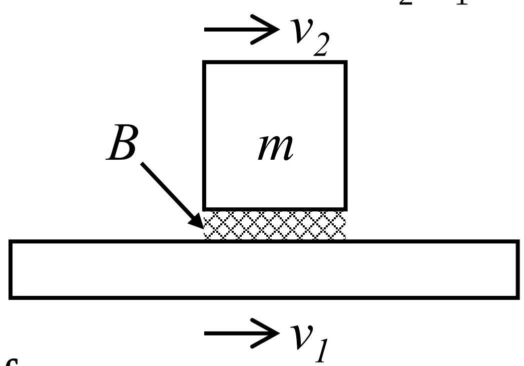


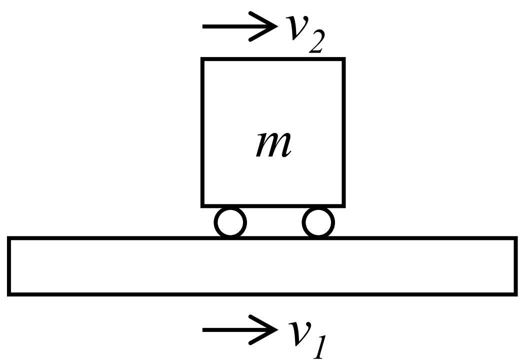
Elements (cont'd)

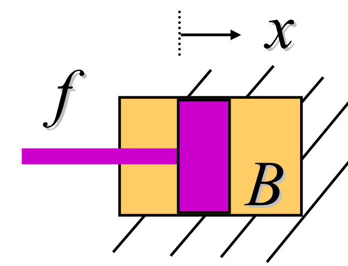
Damping (N-s/m)

Also known as viscous friction or linear friction. Friction is the force that opposes the relative motion or tendency of such motion of two surfaces in contact

$f = B\Delta v$, where $\Delta v = v_2 - v_1$ and B = viscosity constant/coefficient







Above left:

- B is proportional to contact area and viscosity of oil.
- B is inversely proportional to the thickness of film.

Above right:

- B is small enough to be neglected (this is always an approximation.)
- Damping is used to model a dashpot (damper), e.g. shock absorbers on cars.

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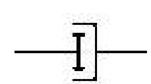
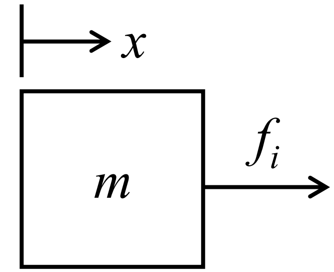
e.g. shock absorbers on cars.



Element Laws

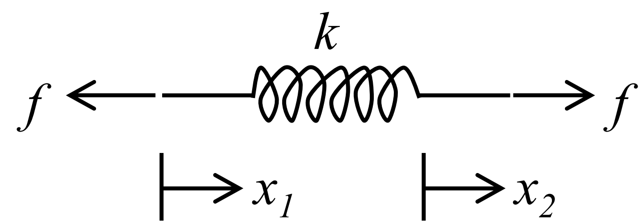
Mass: Newton's 2nd Law

$$ma = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = \sum f_i$$



Stiffness (Spring):

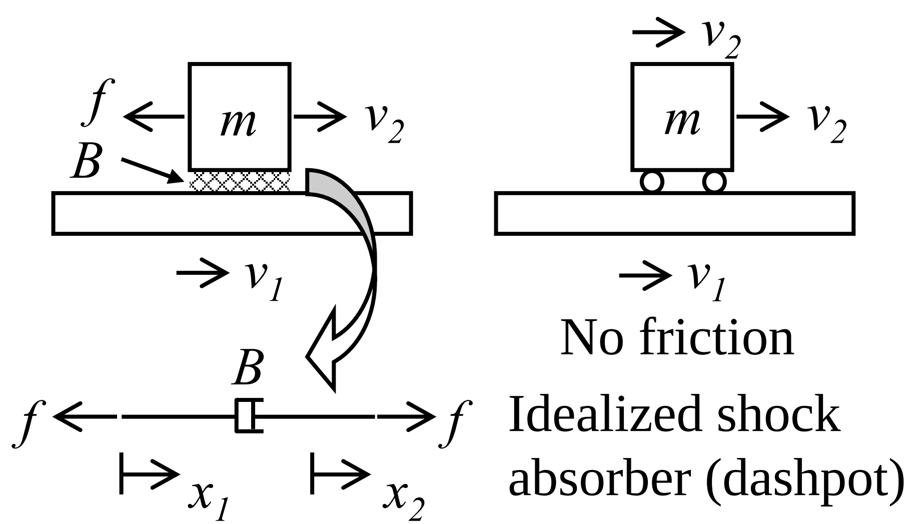
$$f = k(x_2 - x_1)$$



Friction (Damping):

$$f = B(v_2 - v_1) = B\Delta v$$

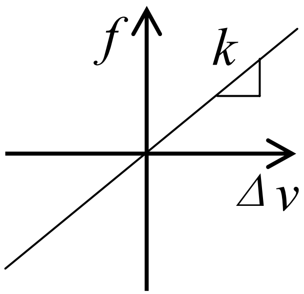
B: viscosity constant,
unit: N-s/m



Element Laws (cont'd)

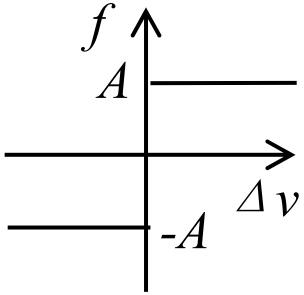
Viscous friction:

$$f = B\Delta v$$



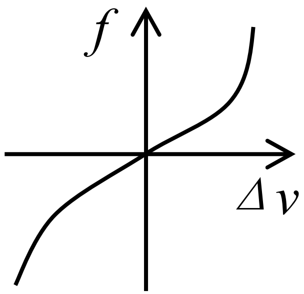
Coulomb (dry) friction:

$$f = A sign(\Delta v)$$



Drag:

$$f = C\Delta x^{a}$$



Interconnection Laws

Determine how to connect elements

- D'Alembert's law
  - Just a re-statement of Newton's 2nd law, summing externally applied forces to a mass

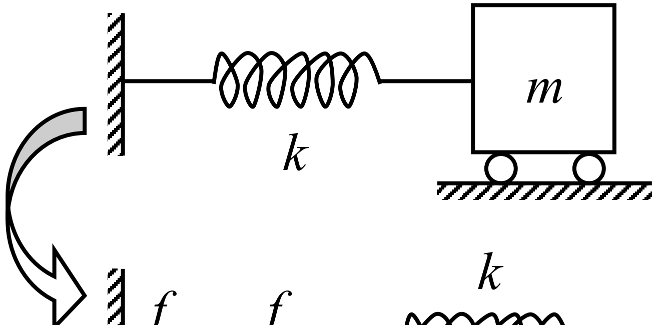
$$\sum_i (f_{ext})_i = ma \quad \text{or} \quad \sum_i (f_{ext})_i - ma = 0$$

  - If you think of $-ma$ as an additional force f_I (the inertial force, or D'Alembert's force), you can then consider it along with all other forces and write

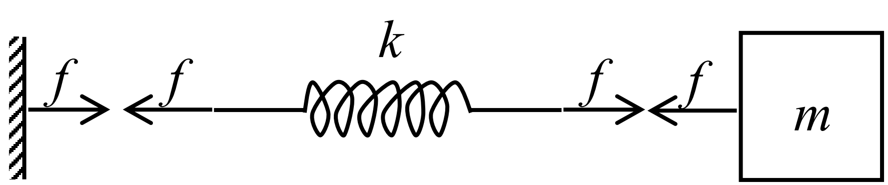
$$\sum f_i = 0$$

- Law of reaction forces (Newton's 3rd law)

  All forces occur in equal and opposite pairs (action/reaction)

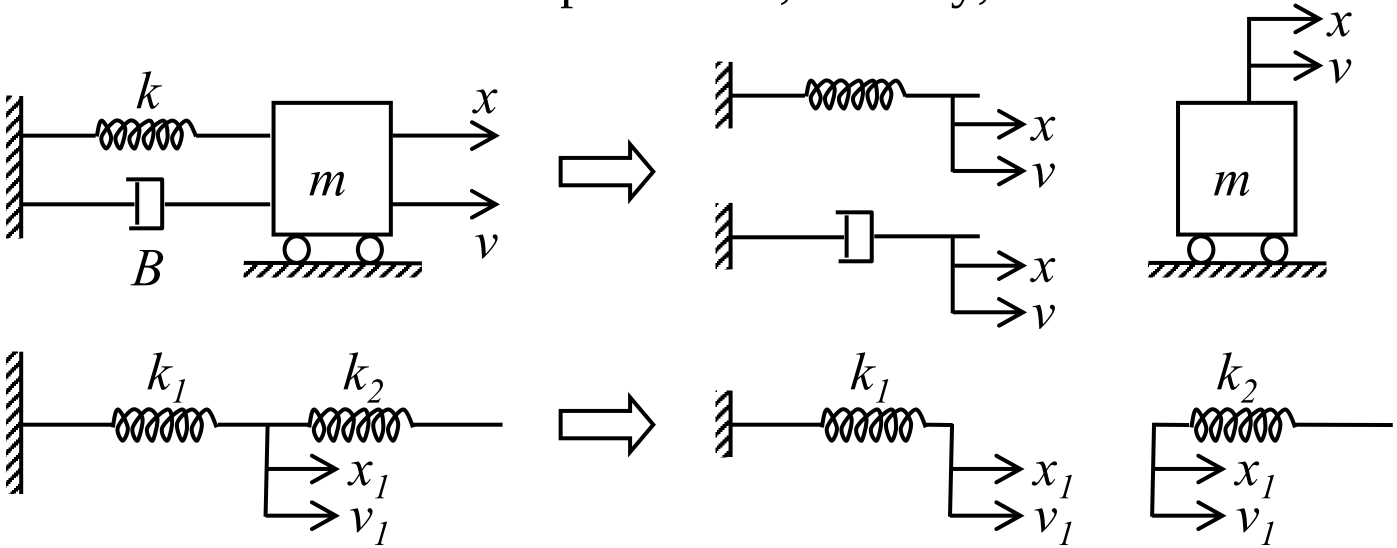


- Force exerted by an element is equal and opposite to the force on the element.

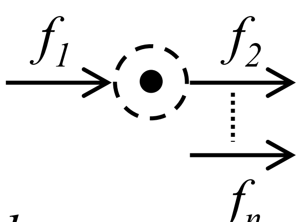


Law of Displacements

– If the ends of two elements are connected, these ends are forced to move with the *same* displacement, velocity, and acceleration.

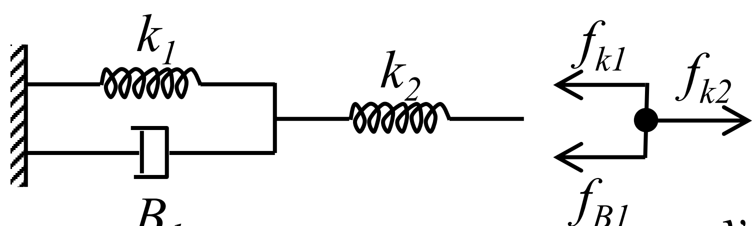


– Newton's 2nd law at a point:
The sum of the forces at a connection between elements equals zero.

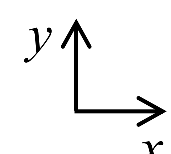


$$\overset{0}{\cancel{m}_P}\,\frac{dv_P}{dt} = \sum f_i \Rightarrow \sum f_i = 0$$

⇑ no mass



$$-f_{k1} - f_{B1} + f_{k2} = 0$$

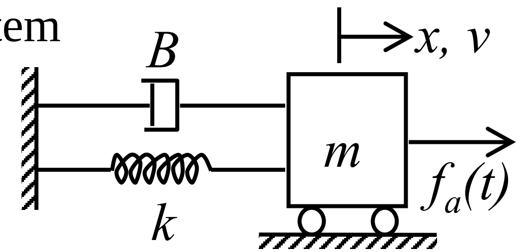


Modeling Steps

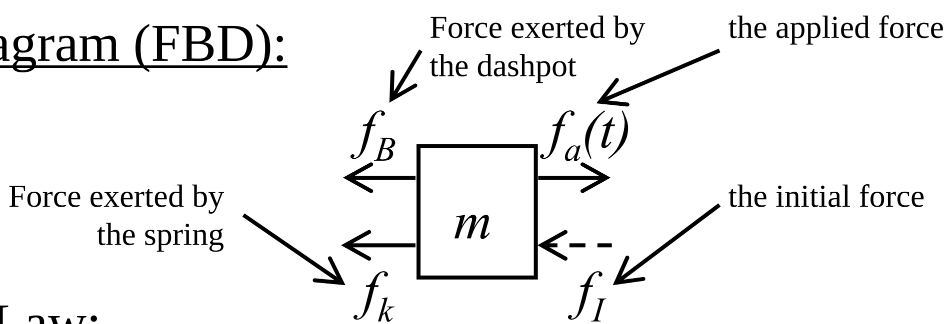
- Understand system function, define problem, and identify input/output variables.
- Draw simplified schematics using basic (idealized) elements.
- Develop mathematical model (differential equations)
  - Identify reference point and positive direction
  - Write elemental equations as well as interconnecting equations by applying physical laws.
  - Draw Free-Body-Diagram (FBD) for each basic element.
  - Combine equations by eliminating intermediate variables.
- Validate model by comparing simulation results with physical measurements.

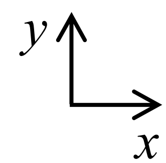
Obtaining the System Model

Example: a mass-spring-damper system



Free Body Diagram (FBD):





Newton's 2nd Law:

$$m\frac{dv}{dt} = f_a(t) - f_B - f_k = f_a(t) - Bv - kx$$

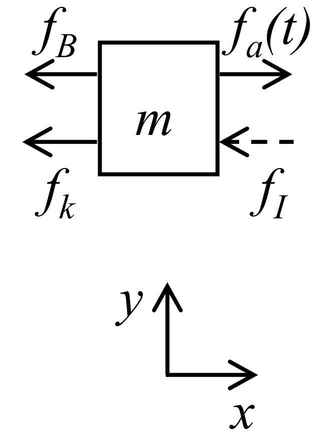
$$m\frac{dv}{dt} + Bv + kx = f_a(t), \quad \frac{dx}{dt} = v$$

$$m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = f_a(t)$$

$$m\ddot{x} + B\dot{x} + kx = f_a(t)$$

Obtaining the System Model (cont'd)

D'Alembert's Law (use the idea of an inertial force, f_I)



$$\sum f_i = 0 = f_a(t) - f_I - f_B - f_k$$
$$= f_a(t) - m\ddot{x} - B\dot{x} - kx = 0$$
$$\Rightarrow m\ddot{x} + B\dot{x} + kx = f_a(t)$$

Obtaining the System Model (cont'd)

Energy distribution:

- EOM of the above simple mass-spring-damper system

$$\underbrace{m\ddot{x}}_{\substack{\text{Contibutin}\\ \text{of Inertia}}} + \underbrace{B\dot{x}}_{\substack{\text{Contribtin}\\ \text{of the Damper}}} + \underbrace{kx}_{\substack{\text{Contribuion}\\ \text{of the Spring}}} = \underbrace{f_a(t)}_{\substack{\text{Total Applied}\\ \text{Force}}}$$

  We now want to look at the energy distribution of the system. How should we do it?

- Multiply the above equation by the velocity v: (since P is defined as $P=fv$)

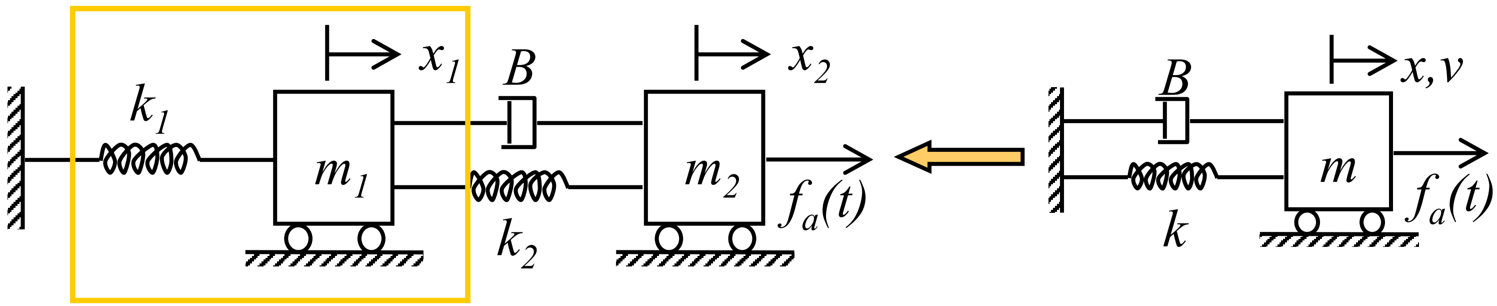
$$m\ddot{x}\cdot\dot{x} + B\dot{x}\cdot\dot{x} + kx\cdot\dot{x} = \underbrace{f_a(t)\cdot\dot{x}}_{\text{Power}}$$

- Integrate the 2nd equation w.r.t. (with respect to) time:

$$\underbrace{\int_{t_0}^{t_1} m\ddot{x}\cdot\dot{x}dt}_{\substack{\Delta E_k\\ \text{change of kenetic}\\ \text{energy}}} + \underbrace{\int_{t_0}^{t_1} B\dot{x}\cdot\dot{x}dt}_{\substack{\int_{t_0}^{t_1} B\dot{x}^2 dt \geq 0\\ \text{Energy dissipated}\\ \text{by damper}}} + \underbrace{\int_{t_0}^{t_1} kx\cdot\dot{x}dt}_{\substack{\Delta E_P\\ \text{Change of}\\ \text{potential energy}}} = \underbrace{\int_{t_0}^{t_1} f_a(t)\cdot\dot{x}dt}_{\substack{W\\ \text{Total work done by}\\ \text{the applied force } f_a(t) \text{ from}\\ \text{time } t_0 \text{ to } t_1}}$$

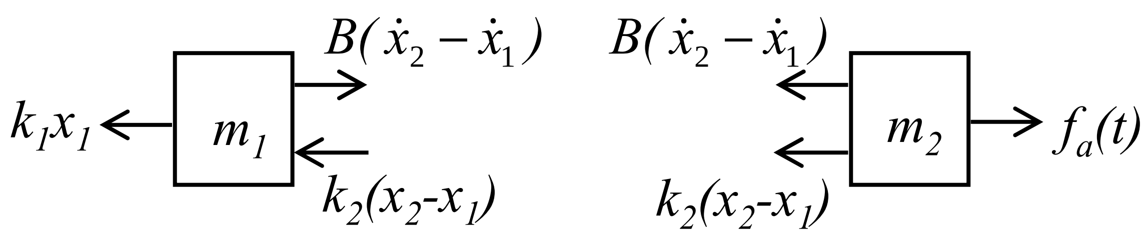
Obtaining the System Model (cont'd)

Example:



Draw the FBDs and write the equations of the system in terms of x_1 and x_2.

Free body diagram (FBD):



Newton's 2nd Law gives:

$$m_1\ddot{x}_1 = B(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) - k_1x_1$$
$$m_2\ddot{x}_2 = f_a(t) - B(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1)$$

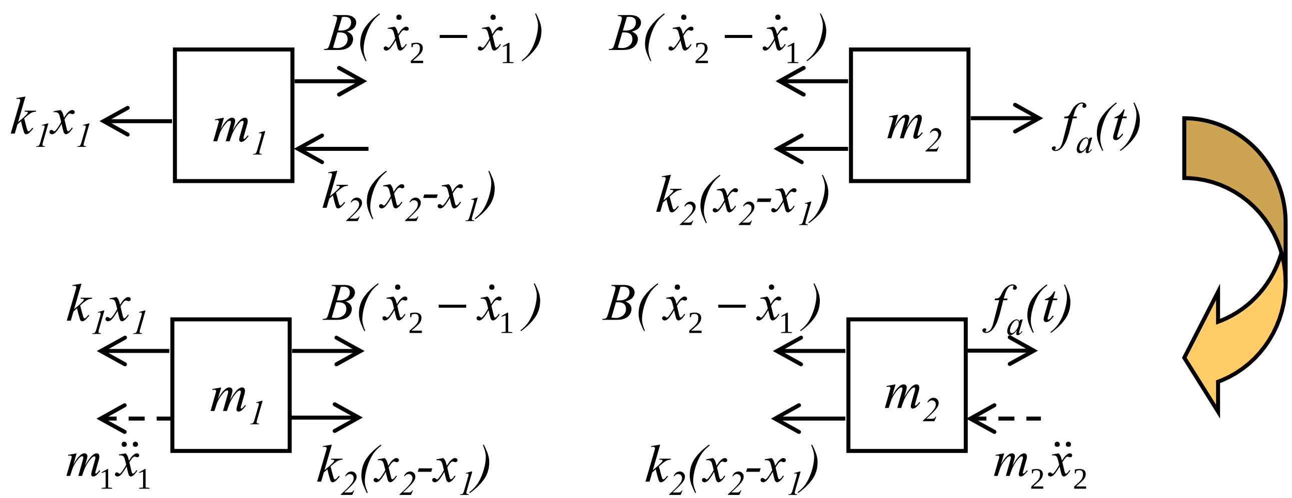
or

$$m_1\ddot{x}_1 - B(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) - k_1x_1 = 0$$
$$m_2\ddot{x}_2 + B(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = f_a(t)$$

Obtaining the System Model (cont'd)

How about deriving the equations using D'Alembert's Law?

Free body diagram (FBD):



$$B(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) - k_1x_1 - m_1\ddot{x}_1 = 0$$

$$f_a(t) - B(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) - m_2\ddot{x}_2 = 0$$

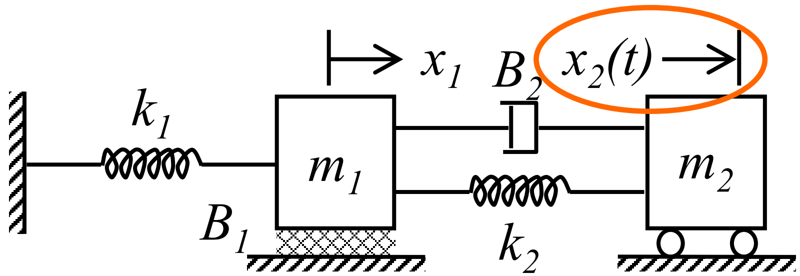
or

$$m_1\ddot{x}_1 - B(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) - k_1x_1 = 0$$

$$m_2\ddot{x}_2 + B(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = f_a(t)$$

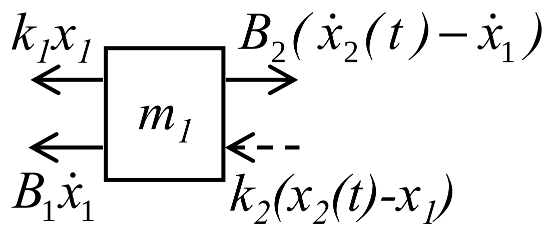
Obtaining the System Model (cont'd)

Example:
Displacement excitation



Draw the FBD and write the equation of the system.

Free body diagram (FBD):



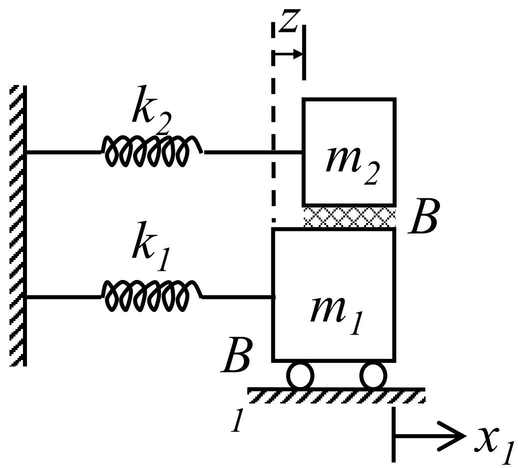
Newton's 2nd Law gives:

$$m_1\ddot{x}_1 = B_2(\dot{x}_2(t)-\dot{x}_1)+k_2(x_2(t)-x_1)-B_1\dot{x}_1-k_1x_1$$

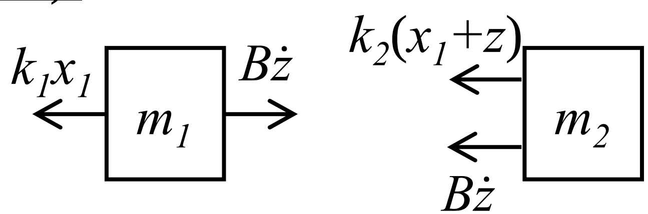
- Do not need an equation for x_2 since it is a defined function $x_2(t)$!

Obtaining the System Model (cont'd)

Example:
Relative displacement



Free body diagram (FBD):



$$x_2 = x_1 + z$$
$$\dot{x}_2 = \dot{x}_1 + \dot{z}$$
$$\ddot{x}_2 = \ddot{x}_1 + \ddot{z}$$

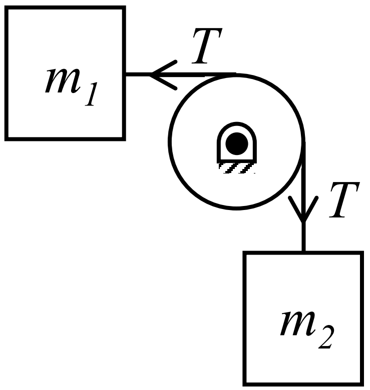
Newton's 2nd Law gives:

$$m_1\ddot{x}_1 = B\dot{z} - k_1x_1$$
$$m_2\ddot{x}_2 = m_2\,(\ddot{x}_1 + \ddot{z}) = -k_2(x_1 + z) - B\dot{z}$$

⇑ Use absolute acceleration

Ideal Pulley Element

Assumption for ideal pulley:



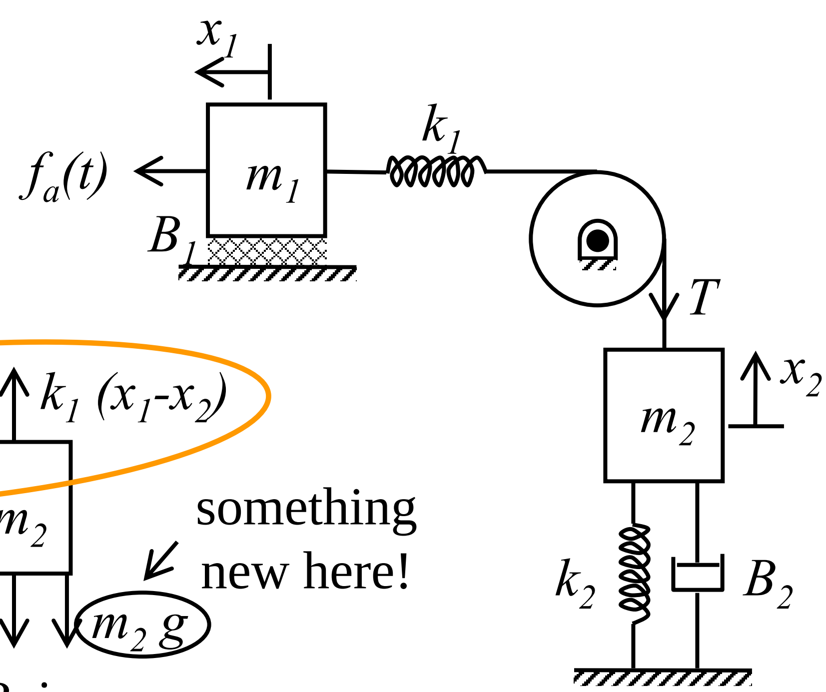
- No mass, no friction, no slippage between cable and cylinder.
- Cable is always in tension.
- Cable cannot stretch.

If the pulley is not ideal, its mass and any frictional effects must be considered.

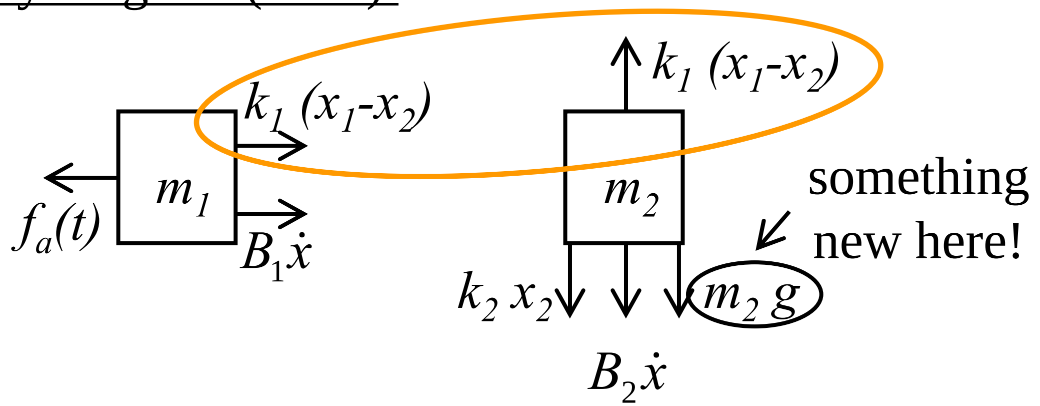
Ideal Pulley Element (cont'd)

Example:

Find the equation describing the system with pulley.



Free body diagram (FBD):



Equations:

$$\text{M1:}\quad -m_1\ddot{x}_1 = -f_a(t) + k_1(x_1 - x_2) + B_1\dot{x}_1$$

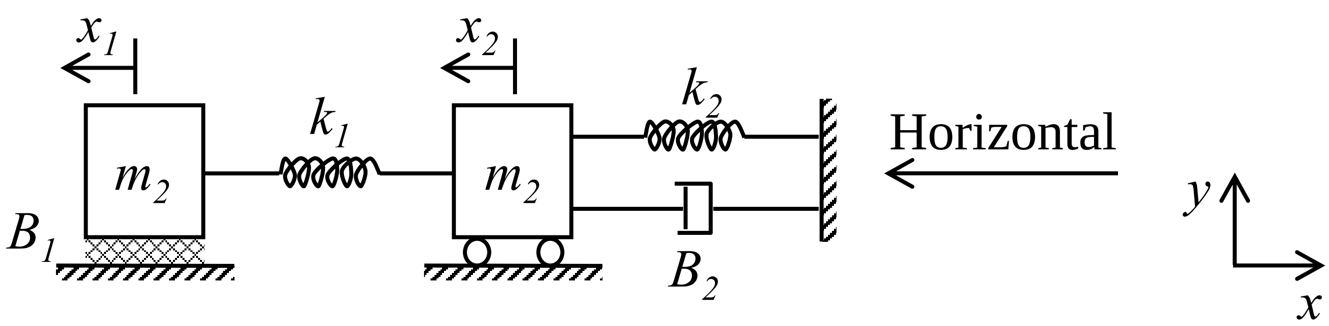
$$\therefore m_1\ddot{x}_1 + B_1\dot{x}_1 + k_1(x_1 - x_2) = f_a(t)$$

$$\text{M2:}\quad m_2\ddot{x}_2 = k_1(x_1 - x_2) - k_2x_2 - B_2\dot{x}_2 - m_2g$$

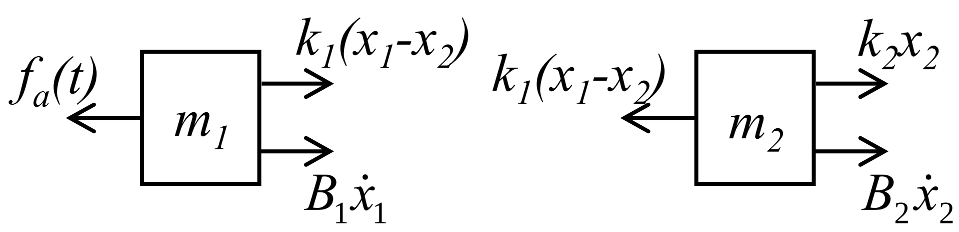
$$\therefore m_2\ddot{x}_2 + B_2\dot{x}_2 + k_2x_2 + m_2g = k_1(x_1 - x_2)$$

Ideal Pulley Element (cont'd)

Compare the above system including an ideal pulley with following system:



Free body diagram (FBD):



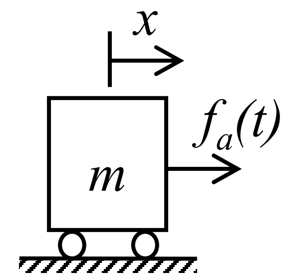
Ideal Pulley Element (cont'd)

Equations for the system:

$$\text{M1}: \quad -m_1\ddot{x}_1 = -f_a(t) + k_1(x_1 - x_2) + B_1\dot{x}_1$$

$$\therefore \quad m_1\ddot{x}_1 + B_1\dot{x}_1 + k_1(x_1 - x_2) = f_a(t)$$

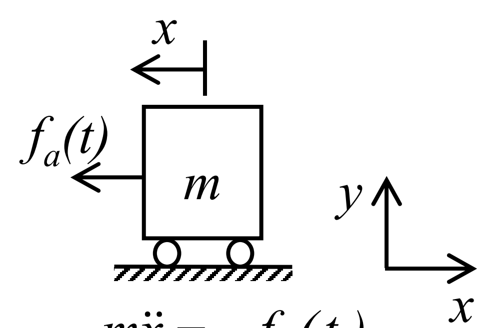
$$\text{or} \quad m_1\dot{v}_1 + B_1 v_1 + k_1(x_1 - x_2) = f_a(t)$$



$$m\ddot{x} = f_a(t)$$

$$\text{M2}: \quad m_2\ddot{x}_2 = k_1(x_1 - x_2) - k_2 x_2 - B_2\dot{x}_2$$

$$\therefore \quad m_2\ddot{x}_2 + B_2\dot{x}_2 + k_2 x_2 = k_1(x_1 - x_2)$$



$$-m\ddot{x} = -f_a(t)$$

$$\therefore m\ddot{x} = f_a(t)$$

Finally,

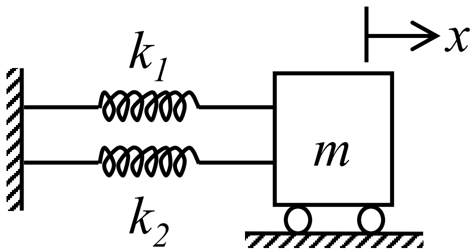
$$\left.\begin{aligned} & m_1\dot{v}_1 + B_1 v_1 + k_1(x_1 - x_2) = f_a(t) \\ & m_2 \underset{\substack{\Uparrow \\ \dot{v}_2}}{\ddot{x}_2} + B_2 \underset{\substack{\Uparrow \\ v_2}}{\dot{x}_2} + k_2 x_2 = k_1(x_1 - x_2) \end{aligned}\right\} \text{(28), p.32}$$

$$\text{M1}: \quad m_1\ddot{x}_1 + B_1\dot{x}_1 + k_1(x_1 - x_2) = f_a(t)$$

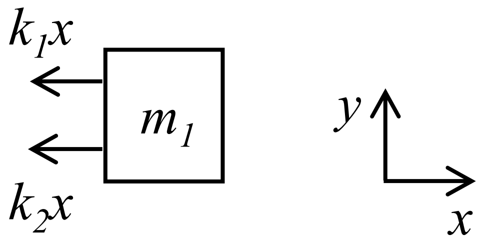
$$\text{M2}: \quad m_2\ddot{x}_2 + B_2\dot{x}_2 + k_2 x_2 + m_2 g = k_1(x_1 - x_2)$$

Parallel Combinations

Parallel Combinations:



FBD:

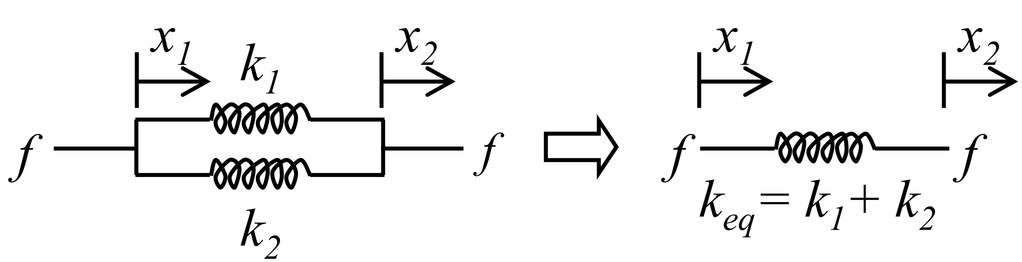


Newton's 2nd law:

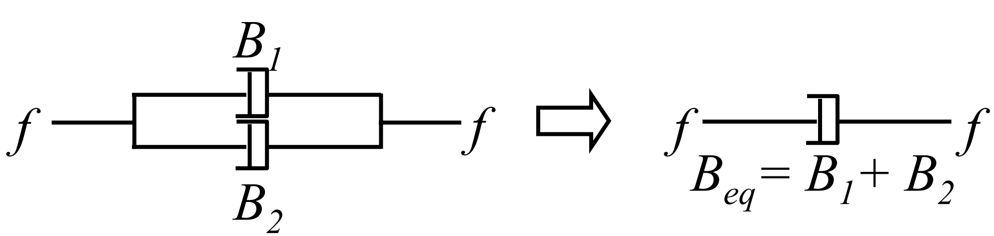
$$m\ddot{x} = -k_1 x - k_2 x \quad \text{or} \quad m\ddot{x} = -(k_1 + k_2)x = -k_{eq}x$$

where, $k_{eq} = k_1 + k_2$, K_{eq} = equivalent spring stiffness

General case:



Similarly,

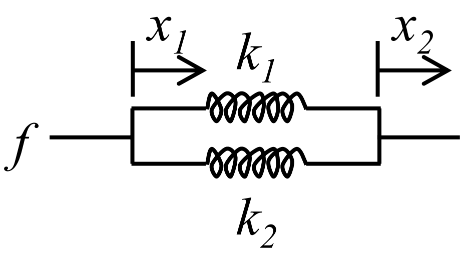


Parallel Combinations (cont'd)

For spring in parallel,

$$f = k_1(x_2 - x_1) + k_2(x_2 - x_1)$$

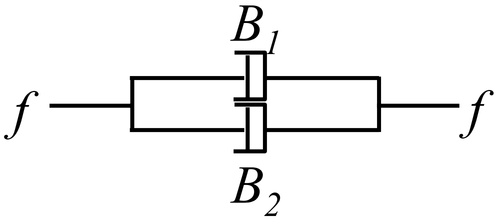
$$\text{or } f = \underbrace{(k_1 + k_2)}_{K_{eq}}(x_2 - x_1)$$



For dampers in parallel,

$$f = B_1(\dot{x}_2 - \dot{x}_1) + B_2(\dot{x}_2 - \dot{x}_1)$$

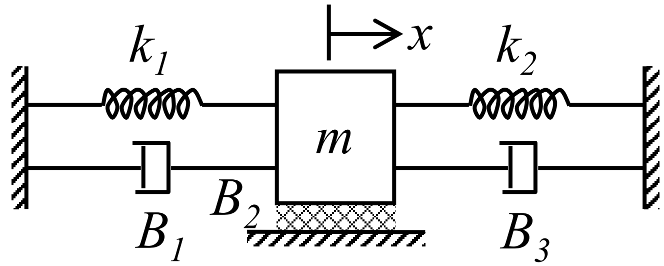
$$\text{or } f = \underbrace{(B_1 + B_2)}_{B_{eq}}(\dot{x}_2 - \dot{x}_1)$$

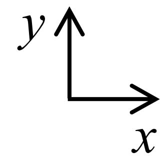


- Elements are in parallel if the first end of each is connected to the same body and the remaining ends are connected to a common body.

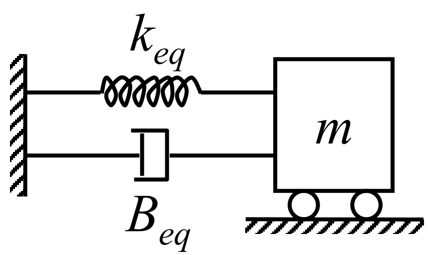
Parallel Combinations (cont'd)

Example:

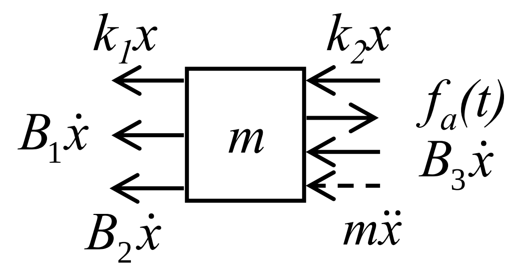




Equivalent system:



FBD:



Equations:

$$m\ddot{x} = f_a(t) - B_1\dot{x} - B_2\dot{x} - B_3\dot{x} - k_1 x - k_2 x$$

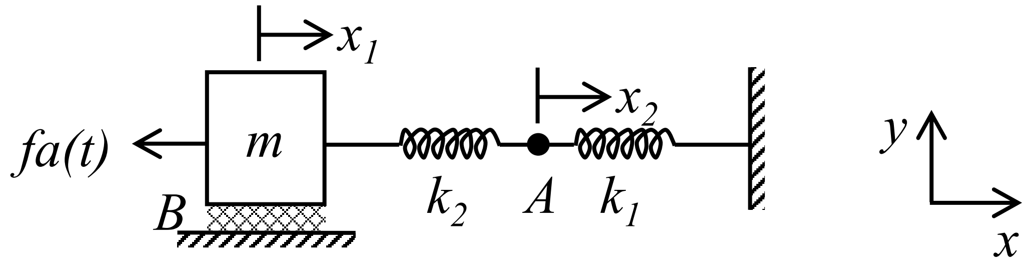
$$= f_a(t) - (\underbrace{B_1 + B_2 + B_3}_{B_{eq}})\dot{x} - (\underbrace{k_1 + k_2}_{k_{eq}})x$$

then, $m\ddot{x} + B_{eq}\dot{x} + k_{eq}x = f_a(t)$

where, $B_{eq} = B_1 + B_2 + B_3$, $k_{\text{eq}} = k_1 + k_2$

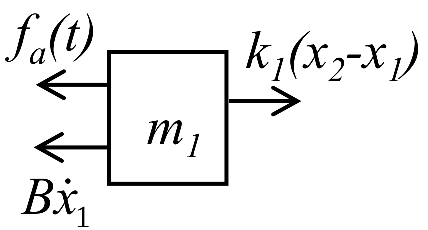
Series Combinations

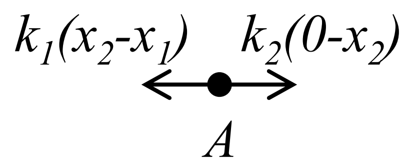
Example:



Draw the FBDs and write an equation only in terms of x_1.

<u>FBD:</u>





Newton's 2nd law for $m: m\ddot{x} = k_1(x_2 - x_1) - B\dot{x}_1 - f_a(t)$ (1)

Newton's 2nd law for $A: -k_1(x_2 - x_1) - k_2 x_2 = 0$

$$\therefore x_2 = \frac{k_1 x_1}{k_1 + k_2} \qquad (2)$$

Series Combinations (cont'd)

Substitute (2) into (1) to get:

$$m\ddot{x} = k_1\left(\frac{k_1 x_1}{k_1 + k_2} - \frac{(k_1 + k_2)x_1}{k_1 + k_2}\right) - B\dot{x}_1 - f_a(t)$$

$$m\ddot{x} = -\underbrace{\frac{k_1 k_2}{k_1 + k_2}}_{k_{eq}} x_1 - B\dot{x}_1 - f_a(t)$$

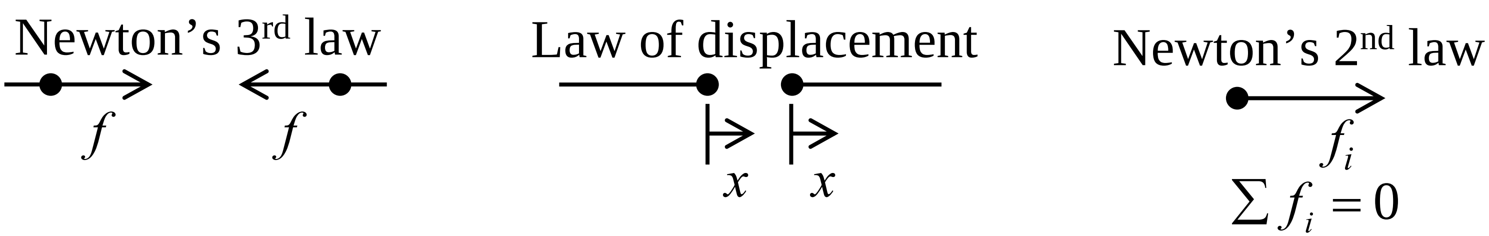
where, $\mathrm{k}_{\mathrm{eq}} = \dfrac{k_1 k_2}{k_1 + k_2} = \dfrac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$

Summary

- Understand the system and identify the elements and variables.
- Divide system into idealized elements: mass, stiffness, friction, pulley.
- Find the element laws:

$$m\ddot{x} = \sum f_i,\ f = k\Delta x,\ f = B\Delta v$$

- Use the interconnection laws:



- Apply Newton's 2nd law to masses, nodes with unknown v and use element, interconnection laws to determine forces in terms of x, v.

Reading and Exercise

- **Reading**
  - Chapter 2
- **Assignment #1**
  - Ex: 2.1, 2.2, 2.16, 2.22, 3.13, 3.28 (to be handed in for marking)

    Due: Fri., 6/7/07 at lecture
  - Ex: 2.10, 2.27, 3.1, 3.15, 3.18 (for your practice, no need to hand in)