Modeling and Simulation of a Three Degree of Freedom Longitudinal Aero plane System

Figure 1: Boeing 777 and example of a two engine business jet
Nonlinear dynamic equations of motion for the longitudinal direction of aircraft

\[ \dot{u} = \frac{X}{m} - g \sin \theta + rv - qw \]
\[ \dot{w} = \frac{Z}{m} + g \cos \phi \cos \theta + qu - pv \]
\[ \dot{x}_1 = (\cos \theta \cos \psi)u + \left( -\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi \right)v + \left( -\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi \right)w \]
\[ \dot{z}_1 = (-\sin \theta)u + (\sin \phi \cos \theta)v + (\cos \phi \cos \theta)w \]
\[ \dot{q} = \left[ M - (I_{xx} - I_{zz})pr - I_{xz} \left( p^2 - r^2 \right) \right] / I_{yy} \]
\[ \dot{\theta} = q \cos \phi - r \sin \phi \]
Nonlinear dynamic equations of motion for the longitudinal direction of aircraft

• The longitudinal equations of motions are considered for the aircraft model with the following assumptions
  • $V=0$
  • $\gamma=0$
  • $p=0$
  • $r=0$
  • $\phi=0$
  • $\psi=0$
  • $I_{xx}=0$
  • $I_{zz}=0$
  • $I_{xz}=0$
Nonlinear dynamic equations of motion for the longitudinal direction of aircraft

The equations after assumptions become:

\[
\begin{align*}
\ddot{u} &= \frac{X}{m} - g \sin \theta - qw \\
\dot{w} &= \frac{Z}{m} + g \cos \theta + qu \\
\dot{x}_1 &= (\cos \theta)u + (\sin \theta)w \\
\dot{z}_1 &= (-\sin \theta)u + (\cos \theta)w \\
\dot{q} &= \frac{[M]}{I_{yy}} \\
\dot{\theta} &= q
\end{align*}
\]
The state vector can be shown as:

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
    x_5 \\
    x_6
\end{bmatrix}
\]

\[= X_{long} =
\begin{bmatrix}
    u \\
    w \\
    x \\
    z \\
    q \\
    \theta
\end{bmatrix}
\]

= \begin{bmatrix}
    \text{Axial Velocity} \\
    \text{Vertical Velocity} \\
    \text{Range} \\
    \text{Altitude}(-) \\
    \text{Pitch Rate} \\
    \text{Pitch Angle}
\end{bmatrix}
The control input can be shown as:

\[
\begin{bmatrix}
u_1 \\ u_2 \\ u_3
\end{bmatrix} = U = \begin{bmatrix}
\delta e \\ \delta t \\ \delta s
\end{bmatrix} = \begin{bmatrix}
\text{Elevator Command} \\
\text{Throttle, \%} \\
\text{stabulator}
\end{bmatrix}
\]
Methods used to stabilize our model

- The Linear Quadratic Regulator (LQR) controller.

- The Proportional Integral Deferential (PID) controller.
Gain Calculations
(Ziegler and Nichols Method)

\[ P \quad \text{control} \quad k_p = 0.5k_{pu} \]

\[ \text{PI control} \quad \begin{cases} 
  k_p = 0.45k_{pu} \\
  k_i = 0.45k_{pu} / (0.83T_u) \end{cases} \]

\[ \text{PID control} \quad \begin{cases} 
  k_p = 0.6k_{pu} \\
  k_i = 0.6k_{pu} / (0.5T_u) \\
  k_d = 0.6k_{pu} / (0.125T_u) \end{cases} \]
Gain Calculations

(Transfer Functions)

\[
\begin{align*}
\frac{X}{\delta e} &= \frac{-7.501e^{-06} s^4 + 2757 s^3 + 4246 s^2 + 281.7 s + 0.002549}{s^6 + 3.531 s^5 + 17.14 s^4 + 2.067 s^3 + 0.1181 s^2 + 1.071e^{-06} s} \\
\frac{Z}{\delta e} &= \frac{-27.83 s^3 - 83.67 s^2 + 3947 s + 67.91}{s^5 + 3.531 s^4 + 17.14 s^3 + 2.067 s^2 + 0.1181 s + 1.071e^{-06}} \\
\frac{\theta}{\delta e} &= \frac{-18.81 s^3 - 29.05 s^2 - 0.6134 s - 1.455e^{-06}}{s^5 + 3.531 s^4 + 17.14 s^3 + 2.067 s^2 + 0.1181 s + 1.071e^{-06}} \\
\frac{X}{\delta T} &= \frac{4.674 s^4 + 16.42 s^3 + 78.44 s^2 - 0.09065 s + 6.172e^{-017}}{s^6 + 3.531 s^5 + 17.14 s^4 + 2.067 s^3 + 0.1181 s^2 + 1.071e^{-06}} \\
\frac{Z}{\delta T} &= \frac{-0.1001 s^3 - 0.7965 s^2 - 2.517 s - 8.463}{s^5 + 3.531 s^4 + 17.14 s^3 + 2.067 s^2 + 0.1181 s + 1.071e^{-06}} \\
\frac{\theta}{\delta T} &= \frac{0.009251 s^2 + 0.05644 s + 5.875e^{-019}}{s^5 + 3.531 s^4 + 17.14 s^3 + 2.067 s^2 + 0.1181 s + 1.071e^{-06}} \\
\frac{X}{\delta S} &= \frac{-1.393e^{-005} s^4 + 4629 s^3 + 7058 s^2 + 468.3 s + 0.004237}{s^6 + 3.531 s^5 + 17.14 s^4 + 2.067 s^3 + 0.1181 s^2 + 1.071e^{-06}} \\
\frac{Z}{\delta S} &= \frac{-52.06 s^3 - 150.4 s^2 + 6553 s + 112.8}{s^5 + 3.531 s^4 + 17.14 s^3 + 2.067 s^2 + 0.1181 s + 1.071e^{-06}} \\
\frac{\theta}{\delta S} &= \frac{-31.58 s^3 - 48.29 s^2 - 1.022 s - 2.444e^{-06}}{s^5 + 3.531 s^4 + 17.14 s^3 + 2.067 s^2 + 0.1181 s + 1.071e^{-06}}
\end{align*}
\]
Root Locus Plots
Root Locus Plots
Root Locus Plots
Cost Function Theory

\[ J = \int_{t_0}^{t} e^2 + \lambda u^2 \, dt \]

\[ e \approx (X - X_{\text{comm}}) \]

\( J \) is the energy spent by the actuators in order to regulate the system towards equilibrium.

The error vector \( (e) \) is defined as the difference between the actual state vector, and the commanded value.

\( \lambda \) is a Lagrange multiplier.
If a different value of weighting is required on each of the elements of the error vector and input $u$:

- A square matrices, denoted here as $Q$, $R$ (identity matrices), are used to ensure that $J$ is non-negative for all values of $e$, and $u$, but is zero when $X$ and $X_{comm}$ (no inputs) are equal.
- For each choice of $Q$, and $R$, minimization of $J$ corresponds to a unique choice of $x$, using specific inputs.
- Essentially, the ratio between $Q$ and $R$ matrices represents the effort on actuators.
Cost Function Theory

Q >> R
The error is penalized, therefore the performances are maximized at the cost of an important effort on the actuators

Q << R
The control effort is penalized, therefore the energy used to compensate is reduced at the cost of lower performances.
Cost Function Theory

\[ Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n} \]

\[ R = \rho \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{r \times r} \]
Highest performance, lowest cost, all state variables, and no inputs were used
lower performance, higher cost, all state variables, no inputs, but integration without dt
RESULTS

Lower performance, higher cost, not all state variables, no inputs, and specific state variables were used for compromise between cost and performance

\[ J_3 = \int_{t_0}^{t} \dot{x}' Q \dot{x} dt \]
RESULTS

\[ J = \int_{t_0}^{t} x'Qx \, dt \]

High performance, higher cost, all state variables, and no inputs were used for compromise between cost and performance.
RESULTS

\[ J_5 = \int_{t_0}^{t} (x'Qx + u' \rho R_{\text{elevator}} u) \, dt \]

High performance, higher cost, all state variables, and elevator input is used only for reducing cost
RESULTS

High performance, higher cost, all state variables, and throttle input is used only for reducing cost
High performance, higher cost, all state variables, and stabilator input is used only for reducing cost.
High performance, highest cost, all state variables, and all inputs were used (which is our real case)
No much difference in results due to $\rho$ that is very small.

- From J1 to J8 (cost is getting higher), and from J1 to J3 (performance is getting better), but from J4 to J8 (no change in performance).

- J8 is our choice for controlling all the inputs with all state variables of the system with high performance and high cost.

- For future work, we can compromise between J3 and J4 to have J9 with specific state variables (not all state variables) and also specific inputs for compromising between performance and cost.
OVERALL SYSTEM LAYOUT

LQR Layout:

Feedback linearization of the model for LQR layout
Dynamic model of longitudinal aircraft for LQR layout
OVERALL SYSTEM LAYOUT

PID Layout:

Dynamic model of longitudinal aircraft for PID layout
OVERALL SYSTEM LAYOUT

PID layout for the longitudinal aircraft model
RESULTS
RESULTS
RESULTS

Body-Axis Component of Inertial Rate

Pitch Rate ($q$, deg/s)

Time, s

Body-Axis Component of Inertial Rate

$q$ (green), deg/s

Time, s
RESULTS

Earth-Relative Aircraft Attitude

Pitch Angle (theta), deg

Time, s

Earth-Relative Aircraft Attitude

theta (green), deg

Time, s
RESULTS

Aerodynamic Angles

Angle of Attack, deg (blue)

Time, s

Flight Path Angles

Vertical (theta-alpha), deg (blue)

Time, s
RESULTS

LQR Feedback Results:

Altitude stabilization

Range stabilization
RESULTS

Pitch angle stabilization
RESULTS

Range stabilization
THANK YOU

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